Math 102
Winter '08
Homework #8.

$\boxed{5.1}$

(4) $\frac{du}{dt} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} u \quad , \quad u(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

Let $u(t) = \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} y \\ e^{\lambda t} z \end{pmatrix}$

Then $\frac{du}{dt} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} u \iff \begin{pmatrix} \lambda e^{\lambda t} y \\ \lambda e^{\lambda t} z \end{pmatrix} = \begin{pmatrix} 1/2\, e^{\lambda t} y + 1/2\, e^{\lambda t} z \\ 1/2\, e^{\lambda t} y + 1/2\, e^{\lambda t} z \end{pmatrix}$.

$\iff \lambda y = 1/2\, y + 1/2\, z$

$\lambda z = 1/2\, y + 1/2\, z$

$\iff \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} u = \lambda u$

So to find λ, look for eigenvalues of $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$:

$\begin{vmatrix} 1/2-\lambda & 1/2 \\ 1/2 & 1/2-\lambda \end{vmatrix} = 0 \iff (1/2-\lambda)^2 - 1/4 = 0$

$\iff \cancel{1/4} - \lambda + \lambda^2 - \cancel{1/4} = 0$

$\iff \lambda(\lambda - 1) = 0 \qquad \iff \lambda = 0, \lambda = 1$

Eigenvector for $\underline{\lambda_1 = 0}$:

$(A - 0I \mid 0) = \left(\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{array}\right) \Rightarrow x_1 + x_2 = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector for $\lambda_1 = 0$

Eigenvector for $\underline{\lambda_2 = 1}$:

$(A - 1I \mid 0) = \left(\begin{array}{cc|c} -1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{array}\right) \rightarrow -x_1 + x_2 = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\therefore \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda_2 = 1$.

$\Rightarrow$ Complete solution: $u(t) = c_1 e^{0t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{1t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$t = 0 \Rightarrow$ need $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 5 \\ -1 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 8 \end{array}\right)$

#4, cont.

$\Rightarrow \begin{matrix} c_1 + c_2 = 5 \\ 2c_2 = 8 \end{matrix} \Rightarrow \boxed{c_2 = 4,\ c_1 = 1}$

So our solution is

$$\boxed{u(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 4e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

or, writing the 2 components of u separately:

$$\boxed{\begin{matrix} v(t) = 1 + 4e^t \\ w(t) = -1 + 4e^t \end{matrix}} \quad \begin{matrix} (v(0) = 5 \checkmark) \\ (w(0) = 3 \checkmark) \end{matrix}$$

⑤ $\lambda(A) = \{3, 1, 0\}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ corr. to $\lambda = 3$

$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ corr. to $\lambda = 1$

$\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ corr. to $\lambda = 0$

$Tr(A) = 4$
$\det(A) = 0$

$\lambda(B) = \{-2, 2, 2\}$

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ corr. to $\lambda = -2$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ corr. to $\lambda = 2$

$Tr(B) = 2$

$Det(A) = 8$.

⑦ λ e'val of A with e'vector x.

(A) WTS: x is also an e'vector of $B = A - 7I$

Pf $Bx = Ax - 7Ix = Ax - 7x = \lambda x - 7x = (\lambda - 7)x$

$\therefore$ x is an e'vector of B corr. to e'value $\lambda - 7$. ☑

(B) Assume $\lambda \neq 0$.

WTS: x is also an e'vector of A^{-1}.

Pf $Ax = \lambda x \Rightarrow A^{-1}(Ax) = A^{-1}(\lambda x)$

$\Rightarrow x = \lambda(A^{-1}x)$

$\Rightarrow \frac{1}{\lambda} x = A^{-1} x$

$\therefore$ x is e'vector of A^{-1} corr. to e'value $1/\lambda$. ☑

(14) • $A = \begin{pmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{pmatrix}$ $\Rightarrow$ rank $A = 1$

$\Rightarrow \dim(\mathrm{Nul}(A)) = 3$

$\Rightarrow$ 0 is an e'val of A with geo. multiplicity 3

$\Rightarrow$ 0 is an e'val of A with algebraic multiplicity 3 or 4.

<u>But</u> $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \doteq \mathrm{Tr}(A) = 4$

$\Rightarrow \lambda_4 = 4.$

$\boxed{\therefore\ \lambda(A) = \{0, 0, 0, 4\}}$

Eigenspace corr. to $\lambda = 4$ is spanned by $\boxed{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$.

• $C = \begin{pmatrix} 0&1&0&1 \\ 1&0&1&0 \\ 0&1&0&1 \\ 1&0&1&0 \end{pmatrix}$ $\Rightarrow$ Rank$(C) = 2$

$\Rightarrow \dim(\mathrm{Nul}\ C) = 2$

$\Rightarrow$ 0 is an e'val of A with geo. multiplicity 2

$\Rightarrow$ 0 is an e'val of A with alg multiplicity 2, 3, or 4

• can't be 3 since eigenvalues must sum to $\mathrm{Tr}(C) = 0$.

$$\begin{pmatrix} 0&1&0&1 \\ 1&0&1&0 \\ 0&1&0&1 \\ 1&0&1&0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} y+w \\ x+z \\ y+w \\ x+z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \\ \lambda w \end{pmatrix} \Rightarrow \begin{matrix} \lambda x = \lambda z \\ \lambda y = \lambda w \end{matrix} \underset{(\lambda \neq 0)}{\Rightarrow} \begin{matrix} x = z \\ y = w. \end{matrix}$$

$\therefore \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of C.

Note $C\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\therefore 2 \in \lambda(C)$

14, cont.

But if 2 is an e'val of C then so is −2 since eigenvalues must add up to 0.

$\boxed{\therefore \lambda(C) = \{2, -2, 0, 0\}}$

(25) $P = uu^T = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{3}{6} \\ \frac{5}{6} \end{pmatrix} \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{3}{6} & \frac{5}{6} \end{pmatrix} = \begin{pmatrix} 1/36 & 1/36 & 3/36 & 5/36 \\ 1/36 & 1/36 & 3/36 & 5/36 \\ 3/36 & 3/36 & 9/36 & 15/36 \\ 5/36 & 5/36 & 15/36 & 25/36 \end{pmatrix}$

(a) $Pu = u\underbrace{u^T u}_{\in \mathbb{R}} = (u^T u)\, u \underset{\substack{\downarrow \\ u \text{ is unit vector}}}{=} 1u = u$

$\therefore$ u is an e'vector of P corr. to $\lambda = 1$.

(b) $v \perp u \Rightarrow v^T u = u^T v = 0$

$\therefore Pv = u\underbrace{u^T v}_{0} = u(0) = 0$

$\Rightarrow \lambda = 0$ is e'val of P with corr. e'vector v.

(c) Find a basis for $u^\perp$ (since by (b), all vectors in $u^\perp$ are e'vectors corr. to $\lambda = 0$):

$X_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} -5 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

(27) $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $\quad |\lambda I - P| = 0 \iff \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{vmatrix} = 0$

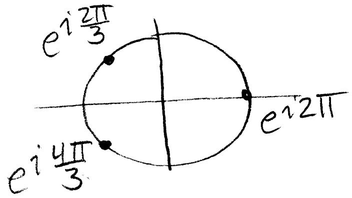
$$\iff \lambda \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -1 & \lambda \end{vmatrix} = 0$$

$$\iff \lambda(\lambda^2) + (0 - (1)) = 0$$

$$\iff \lambda^3 - 1 = 0$$

$$\iff \lambda^3 = 1 = e^{i2\pi}$$

$$\iff \boxed{\lambda = e^{i2\pi/3},\ e^{i4\pi/3},\ e^{i2\pi}}$$



$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\quad |\lambda I - P| = 0 \iff \begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda \end{vmatrix} = 0$

$$\iff \lambda \begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & \lambda-1 \\ -1 & 0 \end{vmatrix} = 0$$

$$\iff \lambda(\lambda-1)(\lambda) - 1(0 - (\lambda-1)(-1)) = 0$$

$$\iff \lambda^2(\lambda-1) - 1(\lambda-1) = 0$$

$$\iff (\lambda^2-1)(\lambda-1) = 0$$

$$\iff (\lambda-1)^2(\lambda+1) = 0$$

$$\iff \boxed{\lambda = 1, 1, -1}$$

$\boxed{5.2}$

(4) A has 3 distinct e'values; $\Lambda = \begin{pmatrix} 1 & & 0 \\ & 2 & \\ 0 & & 7 \end{pmatrix}$.

(5) $\underline{A_1}$: $|A_1 - \lambda I| = 0 \iff \begin{vmatrix} 2-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$\iff -(2-\lambda)^2 - (-4) = 0$

$\iff -(4 - 4\lambda + \lambda^2) + 4 = 0$

$\iff \lambda^2 + 4\lambda = 0 \quad \iff \lambda(\lambda+4) = 0$

$\therefore$ A_1 has distinct e'vals

$\Rightarrow A_1$ diag'ble.

$\underline{A_2}$: distinct e'vals $\{2, -2\} \Rightarrow$ diag'ble.

$\underline{A_3}$: $|A_3 - \lambda I| = 0 \iff \begin{vmatrix} 2-\lambda & 0 \\ 2 & 2-\lambda \end{vmatrix} = 0 \iff (2-\lambda)^2 = 0 \iff \lambda = 2$

Find e'space corr. to $\lambda = 2$:

$[A - 2I \mid 0] = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right) \Rightarrow x_1 = 0,\ x_2$ free

$\Rightarrow$ e'space corr. to $\lambda = 2$ is $\left\{ x \begin{pmatrix} 0 \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}$

$\therefore$ A_3 has only one LI e'vector

$\Rightarrow A_3$ __not__ diag'ble.

(7) $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$: $|A - \lambda I| = 0 \iff \begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\iff (4-\lambda)(2-\lambda) - 3 = 0 \iff 8 - 6\lambda + \lambda^2 - 3 = 0$

$\iff \lambda^2 - 6\lambda + 5 = 0$

$\iff (\lambda - 5)(\lambda - 1) = 0$

$\iff \lambda = 5, 1.$

$\longrightarrow$

#7, cont.

$\lambda=1$ $\quad [A-I\,|\,0] = \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 + x_2 = 0$

$\therefore \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an e'vec. corr. to $\lambda = 1$.

$\lambda=5$ $\quad [A-5I\,|\,0] = \left(\begin{array}{cc|c} -1 & 3 & 0 \\ 1 & -3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow -x_1 + 3x_2 = 0$

$\therefore \begin{pmatrix} 1 \\ 1/3 \end{pmatrix}$ is an e'vec. corr. to $\lambda = 2$.

So $\boxed{A = S^{-1} \Lambda S \text{ where } S = \begin{pmatrix} 1 & 1 \\ -1 & 1/3 \end{pmatrix}, \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}}$

(8) $A = uv^T$.

(A) $Au = u\underbrace{v^T u}_{\text{scalar}} = (v^T u) u$

$\therefore u$ is an e'vector of A corr. to $\lambda = v^T u$.

(B) The other e'values of A are $\boxed{\text{zeros}}$ since rank$(A) = 1$, and $\therefore \dim(\text{Nul}\,A) = n-1$

$\Rightarrow$ geo mult of $\lambda = 0$ is $n-1$

$\Rightarrow$ alg. mult of $\lambda = 0$ is $n-1$ or n

But we already found a nonzero e'val, so $\lambda = 0$ has alg. mult. $n-1$.

(C) $u \in \mathbb{R}^n$, $v \in \mathbb{R}^n \Rightarrow A = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$

$$= \begin{pmatrix} u_1 v_1 & \cdots & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ u_n v_1 & - & - & u_n v_n \end{pmatrix}$$

(i) $\mathrm{Tr}(A) = \sum$ diag entries of A

$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

$= v^T u.$

(ii) $\mathrm{Tr}(A) = \sum$ eigenvalues of A

$= v^T u + 0 + \cdots + 0$

$= v^T u$

Good thing these are the same!

(15) $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{pmatrix}.$

(21) $A^{-1} = \frac{1}{8} \begin{pmatrix} 2 & 0 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ -1/8 & 1/2 \end{pmatrix}$ has e'vals $\lambda = 1/4, 1/2$

$A = \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix}$ has e'vals $\lambda = 4, 2.$

A^{-1} $\quad \lambda = 1/4$: $\left(\begin{array}{cc|c} 0 & 0 & 0 \\ -1/8 & 1/4 & 0 \end{array}\right) \Rightarrow -\frac{1}{8}x_1 + \frac{1}{4}x_2 = 0 \Rightarrow -x_1 + 2x_2 = 0$

$\therefore \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an e'vec of A^{-1} corr. to $\lambda = 1/4$.

$\lambda = 1/2$: $\left(\begin{array}{cc|c} -1/4 & 0 & 0 \\ -1/8 & 0 & 0 \end{array}\right) \Rightarrow x_1 = 0, \; x_2 =$ free

$\therefore \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an e'vec of A^{-1} corr. to $\lambda = 1/2$.

So matrices S that diagonalize A^{-1} have one col. equal to $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and the other equal to $c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

You can check that this is also true of matrices that diagonalize A.

(29) $A^k = S\Lambda^k S^{-1}$ approaches the 0 matrix as $k \to \infty$ iff every λ has $|\lambda| < 1$.

$\lambda(A) = \{.2, 1\}$, so $A^k \not\to 0$ as $k \to \infty$

$\lambda(B) = \{.9, .3\}$, so $B^k \to 0$ as $k \to \infty$.

(30) $(A - .2I \mid 0) = \left(\begin{array}{cc|c} .4 & .4 & 0 \\ .4 & .4 & 0 \end{array}\right) \Rightarrow x_1 + x_2 = 0$

$\therefore \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is e'vector of A corr to $\lambda = .2$

$(A - I \mid 0) = \left(\begin{array}{cc|c} -.4 & .4 & 0 \\ .4 & -.4 & 0 \end{array}\right) \Rightarrow -x_1 + x_2 = 0$

$\therefore \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is e'vector of A corr. to $\lambda = 1$.

So take $\Lambda = \begin{pmatrix} .2 & 0 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow S^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$.

$$\Rightarrow \lim_{k \to \infty} S\Lambda^k S^{-1} = S\left(\lim_{k \to \infty} \Lambda^k\right) S^{-1} = S\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} S^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

(34) If A is diag'ble,

$A = S^{-1}\Lambda S$ where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and $\lambda_1, \ldots, \lambda_n$ are the e'vals of A

$$\begin{aligned} \Rightarrow |A| &= |S^{-1}\Lambda S| \\ &= |S^{-1}||\Lambda||S| \\ &= \frac{1}{\cancel{|S|}} \cdot |\Lambda| \cancel{|S|} \\ &= |\lambda| \\ &= \lambda_1 \cdots \lambda_n \text{ since } \Lambda \text{ is diagonal.} \quad \boxed{\checkmark} \end{aligned}$$

(40) $(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)$

$= (S^{-1}\Lambda S - \lambda_1 I)(S^{-1}\Lambda S - \lambda_2 I)\cdots(S^{-1}\Lambda S - \lambda_n I)$

$= (S^{-1}\Lambda S - \lambda_1 S^{-1} I S)(S^{-1}\Lambda S - \lambda_2 S^{-1} I S)\cdots(S^{-1}\Lambda S - \lambda_n S^{-1} I S)$

$= [S^{-1}(\Lambda - \lambda_1 I)\cancel{S}][\cancel{S^{-1}}(\Lambda - \lambda_2 I)S]\cdots[S^{-1}(\Lambda - \lambda_n I)S]$

$= S^{-1}(\Lambda - \lambda_1 I)(\Lambda - \lambda_2 I)\cdots(\Lambda - \lambda_n I)S$

$$= S^{-1} \underbrace{\begin{pmatrix} 0 & & \\ & \lambda_2 & 0 \\ 0 & & \lambda_n \end{pmatrix}\begin{pmatrix} \lambda_1 & & \\ & 0 & & 0 \\ & & \lambda_3 & \\ 0 & & & 0 \ddots 0 \end{pmatrix}\cdots\begin{pmatrix} \lambda_1 & & \\ & \ddots & 0 \\ 0 & \lambda_{n-1} & 0 \end{pmatrix}} S$$

since $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix}$ can show that this product is the zero matrix

$= 0.$ $\boxed{\checkmark}$

$\boxed{5.3}$

(2) check $A^3 = I$

$A^3 = I \Rightarrow A^6 = I \Rightarrow$ If # of beetles is 3000 at $t=0$, then # of beetles is also 3000 after 6 yrs.

(8) $\underbrace{\begin{pmatrix} d_{k+1} \\ s_{k+1} \\ w_{k+1} \end{pmatrix}}_{u_{k+1}} = \underbrace{\begin{pmatrix} 1 & 1/4 & 0 \\ 0 & 3/4 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} d_k \\ s_k \\ w_k \end{pmatrix}}_{u_k}$ Find steady state.

$$|A-\lambda I|=0 \iff \begin{vmatrix} 1-\lambda & 1/4 & 0 \\ 0 & 3/4-\lambda & 1/2 \\ 0 & 0 & 1/2-\lambda \end{vmatrix} = 0 \iff \lambda = 1/2, 3/4, 1.$$

$\underline{\lambda = 1/2}$: has eigenvector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

$\underline{\lambda = 3/4}$ $\left(\begin{array}{ccc|c} 1/4 & 1/4 & 0 & 0 \\ 0 & 0 & -1/4 & 0 \\ 0 & 0 & -1/4 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 + x_2 = 0 \\ x_3 = 0 \end{array} \Rightarrow \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is e'vec corr to $\lambda = 3/4$.

$\underline{\lambda = 1}$: $\left(\begin{array}{ccc|c} 0 & 1/4 & 0 & 0 \\ 0 & -1/4 & 1/2 & 0 \\ 0 & 0 & -1/2 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_2 = 0 \\ x_3 = 0 \\ x_1 \text{ free} \end{array} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is e'vec corr. to $\lambda=1$.

$\therefore$ $A = S\Lambda S^{-1}$ where $\Lambda = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 3/4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$\Rightarrow S^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

$\boxed{u_k = \underbrace{S\Lambda^k S^{-1}}_{A^k} u_0}$ $\underbrace{\lim_{k\to\infty} A^k u_0}_{u_\infty = \text{steady state}} = S\left(\lim_{k\to\infty} \Lambda^k\right) S^{-1} u_0$

$$= S \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} S^{-1} u_0 = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} d_0 \\ s_0 \\ w_0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} d_0 \\ s_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} d_0 \\ s_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} d_0 + s_0 + w_0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ $u_\infty = \begin{pmatrix} d_\infty \\ s_\infty \\ w_\infty \end{pmatrix} = \begin{pmatrix} d_0 + s_0 + w_0 \\ 0 \\ 0 \end{pmatrix}$. i.e. eventually, everyone is dead :(

(10) $u_k = \begin{pmatrix} y_k \\ z_k \end{pmatrix}$. $A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$, $u_0 = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$.

$\Rightarrow u_k = A^k u_0$

$\boxed{\text{First, diagonalize } A:}$

$\lambda_1 = .5$, $\lambda_2 = 1$

$\underline{\lambda = .5}$ $(A - .5I \mid 0) = \left(\begin{array}{cc|c} .3 & .3 & 0 \\ .2 & .2 & 0 \end{array}\right) \Rightarrow x_1 + x_2 = 0$

$\Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is e'vec corr. to $\lambda = .5$

$\underline{\lambda = 1}$ $(A - I \mid 0) = \left(\begin{array}{cc|c} -.2 & .3 & 0 \\ .2 & -.3 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} -.2 & .3 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow -2x_1 + 3x_2 = 0$

$\Rightarrow \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$ is e'vec. corr. to $\lambda = 1$

So take $\Lambda = \begin{pmatrix} .5 & 0 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 1 \\ -1 & 2/3 \end{pmatrix} \Rightarrow S^{-1} = \begin{pmatrix} .4 & -.6 \\ .6 & .6 \end{pmatrix}$

Now, $u_\infty = \begin{pmatrix} y_\infty \\ z_\infty \end{pmatrix} = \lim_{k \to \infty} \begin{pmatrix} y_k \\ z_k \end{pmatrix}$

$= \lim_{k\to\infty} (A^k u_0) = \lim_{k\to\infty} (S \Lambda^k S^{-1} u_0) = S \left(\lim_{k\to\infty} \Lambda^k\right) S^{-1} u_0$

$= S \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} S^{-1} u_0 = \begin{pmatrix} 1 & 1 \\ -1 & 2/3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} .4 & -.6 \\ .6 & .6 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ -1 & 2/3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2/3 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$\therefore \boxed{\begin{pmatrix} y_\infty \\ z_\infty \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}}$

(12)
$$\begin{pmatrix} B_1 \\ L_1 \\ C_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}}_{A} \begin{pmatrix} B_0 \\ L_0 \\ C_0 \end{pmatrix} \qquad \lambda(A) = \{-1/2, 1/2, 1\}$$

$$\begin{pmatrix} B_\infty \\ L_\infty \\ C_\infty \end{pmatrix} = \lim_{k\to\infty} A^k \begin{pmatrix} B_0 \\ L_0 \\ C_0 \end{pmatrix}.$$

Diagonalize A: $D = \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} -.4082 & -.7071 & -.5774 \\ -.4082 & .7071 & -.5774 \\ 0.8165 & 0 & -.5774 \end{pmatrix}$.

$$\Rightarrow S^{-1} = \begin{pmatrix} -.4082 & -.4082 & .8165 \\ -.7071 & .7071 & 0 \\ -.5774 & .5774 & -.5774 \end{pmatrix}$$

$$\therefore \begin{pmatrix} B_\infty \\ L_\infty \\ C_\infty \end{pmatrix} = S \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} S^{-1} \begin{pmatrix} B_0 \\ L_0 \\ C_0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} B_0 \\ L_0 \\ C_0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 B_0 + 1/3 L_0 + 1/3 C_0 \\ 1/3 B_0 + 1/3 L_0 + 1/3 C_0 \\ 1/3 B_0 + 1/3 L_0 + 1/3 C_0 \end{pmatrix}.$$

(15) Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ be a Markov matrix, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Then $AX = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n \end{pmatrix}$.

$\therefore$ sum of components of $AX =$

$(a_{11}x_1 + \cdots + a_{1n}x_n) + \cdots + (a_{n1}x_1 + \cdots + a_{nn}x_n)$

$= (a_{11} + \cdots + a_{n1})x_1 + \cdots + (a_{1n} + \cdots + a_{nn})x_n$

$= 1x_1 + \cdots + 1x_n$ (since A is Markov).

$=$ sum of components of X

#15, cont.

b) Let $Ax = \lambda x$ with $\lambda \neq 1$

Then by (a), sum of components of Ax =

sum of components of λx

= sum of components of x.

$\therefore$ λ (sum of comps of x) = (sum of comps. of x)

$\Rightarrow$ (sum of comps. of x) = 0 ☑

(25) (a) Diagonalize A: $S = \begin{pmatrix} -.7071 & .7071 \\ .7071 & .7071 \end{pmatrix}$

$D = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$

Then $A^{1/2} = SD^{1/2}S^{-1}$

$\Rightarrow S\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}S^{-1}$ is a matrix sqrt. of A.

(b) $\sqrt{B}$ would have e'values $\sqrt{9}$ and $\sqrt{-1}$

(28) (a) Always: Nul A = eigenspace corr. to $\lambda = 0$.

(b) When the sum of the geometric multiplicities of the λ's is equal to the rank of A.

In particular, this happens if A is diag'ble.

5.4

① $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ $\lambda(A) = \{-2, 0\}$

$\underline{\lambda = -2}$ $[A + 2I \mid 0] = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \Rightarrow x_1 + x_2 = 0 \Rightarrow \boxed{X_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is e'vector

corr. to $\lambda = -2$

$\underline{\lambda = 0}$ $\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to -x_1 + x_2 = 0 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boxed{X_2}$ is e'vector

corr to $\lambda = 0$.

$\Rightarrow S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \Lambda = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow S^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$.

$= e^{At} = S e^{\Lambda t} S^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & \cancel{e^{0t}}\,1 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$

$= \begin{pmatrix} e^{-2t} & 1 \\ -e^{-2t} & 1 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$

$= \begin{pmatrix} 1/2\, e^{-2t} + 1/2 & -1/2\, e^{-2t} + 1/2 \\ -1/2\, e^{-2t} + 1/2 & 1/2\, e^{-2t} + 1/2 \end{pmatrix}$

$= \boxed{\frac{1}{2} \begin{pmatrix} e^{-2t} + 1 & -e^{-2t} + 1 \\ -e^{-2t} + 1 & e^{-2t} + 1 \end{pmatrix}}$

② ⓐ $\frac{du}{dt} = Au$ has general solution

$$\boxed{u = \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} e^{-2t} & 0 \\ 0 & 1 \end{pmatrix}}_{e^{\Lambda t}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$$

equivalent

$= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-2t} \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} + c_2 \\ -c_1 e^{-2t} + c_2 \end{pmatrix}$

$\boxed{\therefore\ u(t) = c_1 e^{-2t} X_1 + c_2 X_2}$

#2, cont.

(b) $u(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\Rightarrow c_1 = 1,\ c_2 = 2$

$\boxed{\therefore\ u(t) = e^{-2t} x_1 + 2x_2}$

(c) As $t \to \infty$, $e^{\Lambda t} = \begin{pmatrix} e^{-2t} & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow u(t) \to \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}$

(3) $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \Rightarrow \lambda(A) = \{0, 2\}$

Find S: $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 - x_2 = 0$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is e'vector corr. to $\lambda = 0$.

$(A - 2I \mid 0) = \left(\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 + x_2 = 0$

$\Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is e'vector corr. to $\lambda = 2$.

$\therefore\ \frac{du}{dt} = Au$ has gen. sol $\boxed{u = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{0t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$

$u(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Rightarrow \boxed{u(t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

As $t \to \infty$, $e^{2t} \to \infty$ so $u(t)$ blows up.

⑤ ⓐ $e^{A(t+T)} = S e^{\Lambda(t+T)} S^{-1}$

$= S e^{\Lambda t} e^{\Lambda T} S^{-1}$

$= S e^{\Lambda t} \underbrace{S^{-1} S}_{I} e^{\Lambda T} S^{-1}$

$= e^{At} \cdot e^{AT}$ ☑

ⓑ (see back of book).

⑧ $\begin{pmatrix} r'(t) \\ w'(t) \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} r(t) \\ w(t) \end{pmatrix}$

"$\frac{du}{dt}$" "A" "u"

ⓐ $\lambda(A) = \{3, 2\} \Rightarrow$ system unstable.

ⓑ Find S, Λ:

$(A - 3I \mid 0) = \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 - 2x_2 = 0$

$\Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is e'vector corr. to $\lambda = 3$.

$(A - 2I \mid 0) = \left(\begin{array}{cc|c} 2 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 - x_2 = 0$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is e'vector corr. to $\lambda = 2$

So $S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

$\Rightarrow \frac{du}{dt} = Au$ has gen. sol. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = u(t)$.

If $u(0) = \begin{pmatrix} 300 \\ 200 \end{pmatrix}$, get $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$

$\therefore \boxed{u(t) = \begin{pmatrix} r(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 100 \\ 100 \end{pmatrix}}$

#8, cont.

ⓒ $\begin{pmatrix} r(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 100e^{3t} \\ 100e^{2t} \end{pmatrix} = \begin{pmatrix} 200e^{3t} + 100e^{2t} \\ 100e^{3t} + 100e^{2t} \end{pmatrix}$

$\therefore$ as $t \to \infty$, $\begin{pmatrix} r(t) \\ w(t) \end{pmatrix} \to \begin{pmatrix} 200e^{3t} \\ 100e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} 200 \\ 100 \end{pmatrix}$.

$\Rightarrow$ after a long time, rabbits : wolves $\approx 2:1$.

⑨ see back of book.