Math 102
Winter '08
Homework #7

4.3

(3) a) True: $|S^{-1}AS| = |S^{-1}||A||S| = \frac{1}{|S|}|A|\cdot|S| = |A|$

b) False: Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $|A| = 0$ but none of the cofactors are 0

c) False: Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow |A| = 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1(1) - 1(-1)$
$= 2$

(5) Let $F_n = \begin{vmatrix} 1 & -1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & 1 & 1 \end{vmatrix}$ Tridiagonal matrix.

Claim: $\{F_n\}$ is the Fibonacci sequence!

Pf: $F_1 = |1|$, $F_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 - (-1)(1) = 2$.

so the 1st two terms match up. So, it remains to show that $\{F_n\}$ satisfies the rule:

$$F_n = F_{n-1} + F_{n-2}$$

$$F_n = \underbrace{\begin{vmatrix} 1 & -1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & 1 & 1 \end{vmatrix}}_{(n)} (n) = 1 \underbrace{\begin{vmatrix} 1 & -1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & 1 & 1 \end{vmatrix}}_{(n-1)} (n-1) - (-1) \underbrace{\begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & -1 & 0 \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & 1 & 1 \end{vmatrix}}_{(n-1)} (n-1)$$

by cofactor exp. along first row

cofactor exp. of 2nd determinant along first column

$$= 1 \cdot F_{n-1} + 1\left(1 \cdot \underbrace{\begin{vmatrix} 1 & -1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & 1 & 1 \end{vmatrix}}_{(n-2)} (n-2)\right)$$

$$= F_{n-1} + F_{n-2}.$$

(28) a) $C_1 = |0| = 0$

$$C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

$$C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1\, C_2 = 1$$

b) In general, $\boxed{C_n = -C_{n-2}}$

$\therefore C_{10} = -C_8 = -(-C_6) = -(-(-C_4)) = -(-(-1)) = \boxed{-1}$.

(34) a) <u>Claim</u>: $\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A|\,|D|$

"<u>proof</u>:" Assume A, B, C, D all $n \times n$ (so that $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is $(2n) \times (2n)$).

To find $\begin{vmatrix} A & B \\ 0 & D \end{vmatrix}$, we could do $2n$ steps of Gaussian elimination (GE) until we have an upper Δ matrix, and then we'd compute $\begin{vmatrix} A & B \\ 0 & D \end{vmatrix}$ by multiplying the diagonal entries.

<u>However, because of the zero block in the lower left corner</u>, this is the same as doing n steps of GE on A, followed by n steps of GE on D! We get the same diagonal entries either way, so

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A| \cdot |D|.$$

☑

#34, cont.

b)
$$\begin{array}{c} \\ A \\ \\ C \end{array} \begin{vmatrix} 1 & 2 & -1 & -5 \\ 6 & 3 & 2 & 6 \\ 3 & 4 & 1 & 7 \\ 3 & -2 & -5 & 3 \end{vmatrix} \begin{array}{c} B \\ \\ \\ D \end{array} = -868$$

but $|A||D| - |B||C| = -270 \neq -868$.

<u>Note</u>: MATLAB computes determinants!
use command: det(A).

c) Same counterexample as in (b):
$\det(AD - CB) = -1084 \neq -868$.

(36) $\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$ (assuming A^{-1} exists!)

$\Rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$

Taking determinants of both sides yields:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} I & 0 \\ CA^{-1} & I \end{vmatrix}\begin{vmatrix} A & B \\ 0 & D-CA^{-1}B \end{vmatrix}$$

$= 1 \cdot \begin{vmatrix} A & B \\ 0 & D-CA^{-1}B \end{vmatrix}$ because product of diag. entries of $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}$ is 1.

$= |A| \cdot |D-CA^{-1}B|$ by #34 (a)

For the second equality, note that if $AC = CA$

$$|A| \cdot |D-CA^{-1}B| = |A(D-CA^{-1}B)|$$
$$= |AD - \underbrace{ACA^{-1}}B|$$
$$= |AD - CB|$$ since $ACA^{-1} = C$ $\boxed{\checkmark}$

Let P_n denote $n \times n$ Pascal matrix.

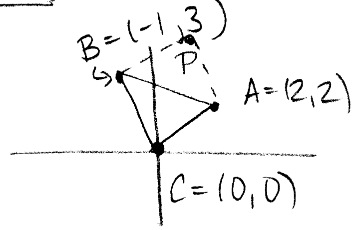
(43) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 19 \end{vmatrix} = -1\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 6 & 10 \end{vmatrix} - 10\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 10 \end{vmatrix} + (20-1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}$

$$= \underbrace{-1\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 6 & 10 \end{vmatrix} - 10\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 10 \end{vmatrix} + 20\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}}_{= \det(P_4) = 1} - \underbrace{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}}_{= \det(P_3) = 1}$$

$= 1 - 1 = 0$ $\boxed{\checkmark}$

4.4

5 a)



Note: Area of parallelogram ABCP

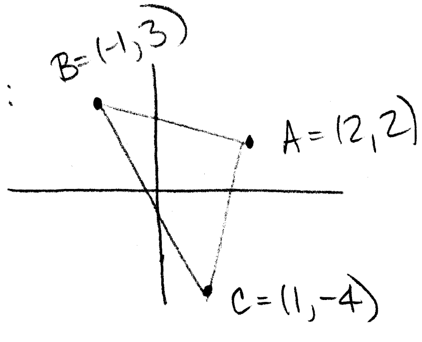
$$= \begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix}.$$

Area of Δ ABC $= \frac{1}{2}$(area of ||'ogram ABCP)

$$= \frac{1}{2}\left|\begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix}\right|$$

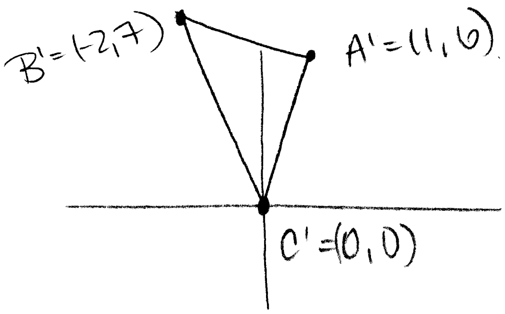
b)

Area of this Δ:



(call this triangle T)

is the same as the area of the translated Δ:



$A' = A - C = (2,2) - (1,-4) = (1,6)$

$B' = B - C = (-1,3) - (1,-4) = (-2,7)$

$C' = C - C = (0,0)$

and the area of this Δ is $\begin{vmatrix} 1 & 6 \\ -2 & 7 \end{vmatrix}$.

$$\therefore \left|\begin{vmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -4 & 1 \end{vmatrix}\right| \underset{\substack{R1-R3 \\ R2-R3}}{=} \begin{vmatrix} 1 & 6 & 0 \\ -2 & 7 & 0 \\ 1 & -4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 6 \\ -2 & 7 \end{vmatrix} = \text{area of } T.$$

cofactor exp along 3rd column

This has the effect of shifting A to A' and B to B'!

(6) Let A be an $n \times n$ matrix.

$\text{Det}(3A) =$ Vol of parallelopiped with side lengths 3(row 1 of A), 3(row 2 of A), ..., 3(row n of A)

$= 3^n \cdot$ [Vol of parallelopiped with side lengths (row 1 of A), ..., (row n of A)]

$= 3^n \cdot \det(A)$.

example. In $\mathbb{R}^3$, if we start with the unit cube and multiply each side length by 3, the volume of the resulting cube is

$$\begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 3^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3^3 \cdot (\text{vol. of unit cube})$$

(10) P an odd permutation

$\Rightarrow \det P = -1$

$\Rightarrow \det(P^2) = (\det P)^2 = (-1)^2 = 1$

$\Rightarrow P^2$ is even. $\boxed{\checkmark}$

Also, $P \cdot P^{-1} = I$

$\Rightarrow \det(P \cdot P^{-1}) = \det(I)$

$\Rightarrow \det P \cdot \det P^{-1} = 1$

$\Rightarrow -1 \cdot \det(P^{-1}) = 1$

$\Rightarrow \det(P^{-1}) = -1$

$\Rightarrow P^{-1}$ odd. $\boxed{\checkmark}$

(14) (a) $ax+by=1$, $cx+dy=0$ $\Rightarrow \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$y = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{0-c}{ad-bc} = \boxed{\frac{-c}{ad-bc}}$$

b) $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

By Cramer's Rule, $y = \dfrac{\begin{vmatrix} a & 1 & c \\ d & 0 & f \\ g & 0 & i \end{vmatrix}}{D} = \dfrac{-\begin{vmatrix} d & f \\ g & i \end{vmatrix}}{D} = \boxed{\dfrac{-(di-fg)}{D}}$

(16) a) $|\hat{b}\ \hat{a}_2\ \hat{a}_3| = x_1 \det A \Rightarrow x_1 = \dfrac{|\hat{b}\ \hat{a}_2\ \hat{a}_3|}{\det A} = \dfrac{\det B_1}{\det A}$

b) $x_1|\hat{a}_1\ \hat{a}_2\ \hat{a}_3| = |x_1\hat{a}_1\ \hat{a}_2\ \hat{a}_3|$ since multiplying a col. by x_1 multiplies the determinant by x_1

$= |x_1\hat{a}_1 + x_2\hat{a}_2 + x_3\hat{a}_3\ \ \hat{a}_2\ \hat{a}_3|$ since adding multiples of cols 2 and 3 to col 1 leaves the determinant unchanged.

(18) a) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{pmatrix} \Rightarrow C = \begin{pmatrix} \begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix} \\ -\begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 1 & -4 \\ 0 & 0 & 3 \end{pmatrix}$

$\therefore A^{-1} = \dfrac{C^T}{|A|} = \dfrac{1}{3}\begin{pmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & -4/3 & 1 \end{pmatrix}$

b) Same process as (a)

$$A^{-1} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix}$$

(A is 4x4)

(23) $A^{-1} = \frac{C^T}{\det A} \Rightarrow A \cdot A^{-1} = \frac{A C^T}{\det A} \Rightarrow (\det A) I = A C^T$

$\Rightarrow \det[(\det A) \cdot I] = \det(A) \cdot \det(C^T)$

$$\Rightarrow \underbrace{\begin{vmatrix} \det A & & & 0 \\ & \det A & & \\ & & \det A & \\ 0 & & & \det A \end{vmatrix}}_{[\det(A)]^4} = \det(A) \det(C)$$

$\Rightarrow \det(C) = [\det(A)]^3$

$\Rightarrow \det A = (\det C)^{1/3}$. Since we know all the cofactors, we know C and can compute det C. Thus we can also compute det A.

Recall $A^{-1} = \frac{C^T}{\det A}$. So, compute A^{-1}, and then invert it to find A!

(31) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} = 16 = $ vol. of $\underbrace{\text{hypercube in } \mathbb{R}^4}_{}$

4-dimensional analogue of square ($n=2$) and cube ($n=3$)

(37) $J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

length of col 1 $= \sqrt{\cos^2\theta + \sin^2\theta} = 1$

length of col 2 $= \sqrt{r^2\sin^2\theta + r^2\cos^2\theta} = r$

cols are orthogonal since

$(\cos\theta \;\; \sin\theta)\begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix} = -r\cos\theta\sin\theta + r\cos\theta\sin\theta = 0$

$\therefore J = 1 \times r = \boxed{r}$