Math 102
Winter '08
Homework # 6

$\boxed{3.4}$

⑬ $A = \begin{array}{c} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ a & b & c \end{array} \end{array}$

- $\|a\| = \sqrt{0^2+0^2+1^2} = 1$ ✓, so let $\boxed{q_1 = a = \begin{pmatrix}0\\0\\1\end{pmatrix}}$
- $u_2 = b - \text{proj}_{q_1} b = \begin{pmatrix}0\\1\\1\end{pmatrix} - \left[(0\ 1\ 1)\begin{pmatrix}0\\0\\1\end{pmatrix}\right]\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$= \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \Rightarrow \|u_2\| = 1, \text{ so } \boxed{q_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}}$$

- $u_3 = c - \text{proj}_{q_1} c - \text{proj}_{q_2} c$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} - \left[(1\ 1\ 1)\begin{pmatrix}0\\0\\1\end{pmatrix}\right]\begin{pmatrix}0\\0\\1\end{pmatrix} - \left[(1\ 1\ 1)\begin{pmatrix}0\\1\\0\end{pmatrix}\right]\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow \|u_3\| = 1, \text{ so}$$

$$\boxed{q_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}}$$

$$Q = \left(q_1 \,|\, q_2 \,|\, q_3\right) = \boxed{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}$$

$$R = \begin{pmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & q_3^T c \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}$$

check $QR = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = A$ ✓

⑮ $A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix}$ Find o.n.b. for Col(A):

- $a_1 = \begin{pmatrix}1\\2\\-2\end{pmatrix}$ $\|a_1\| = \sqrt{1^2+2^2+(-2)^2} = 3 \Rightarrow \boxed{q_1 = \begin{pmatrix}1/3\\2/3\\-2/3\end{pmatrix}}$
- $u_2 = a_2 - \text{proj}_{q_1} a_2 = \begin{pmatrix}1\\-1\\4\end{pmatrix} - \left[(1\ -1\ 4)\begin{pmatrix}1/3\\2/3\\-2/3\end{pmatrix}\right]\begin{pmatrix}1/3\\2/3\\-2/3\end{pmatrix}$

$$= \begin{pmatrix}1\\-1\\4\end{pmatrix} - \left[1/3 - 2/3 - 8/3\right]\begin{pmatrix}1/3\\2/3\\-2/3\end{pmatrix} = \begin{pmatrix}1\\-1\\4\end{pmatrix} + 3\begin{pmatrix}1/3\\2/3\\-2/3\end{pmatrix} = \begin{pmatrix}2\\1\\2\end{pmatrix}$$

$$\|u_2\| = \sqrt{2^2+1^2+2^2} = 3 \Rightarrow \boxed{q_2 = \begin{pmatrix}2/3\\1/3\\2/3\end{pmatrix}}$$

3.4 cont.

Let $a_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ $\quad u_3 = a_3 - \text{proj}_{q_1} a_3 - \text{proj}_{q_2} a_3$

$$= \begin{pmatrix}0\\0\\1\end{pmatrix} - \left[(0\ 0\ 1)\begin{pmatrix}1/3\\2/3\\-2/3\end{pmatrix}\right]\begin{pmatrix}1/3\\2/3\\-2/3\end{pmatrix} - \left[(0\ 0\ 1)\begin{pmatrix}2/3\\1/3\\2/3\end{pmatrix}\right]\begin{pmatrix}2/3\\1/3\\2/3\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\1\end{pmatrix} + 2/3\begin{pmatrix}1/3\\2/3\\-2/3\end{pmatrix} - 2/3\begin{pmatrix}2/3\\1/3\\2/3\end{pmatrix} = \begin{pmatrix}-2/9\\2/9\\-8/9\end{pmatrix}$$

$$\|u_3\| = \sqrt{4/81 + 4/81 + 64/81} = = 6\sqrt{2}/9$$

$$\Rightarrow \boxed{q_3 = u_3 / (6\sqrt{2}/9)}$$

$$\boxed{q_3 \in (\text{Col } A)^{\perp} = \text{Nul}(A^T)}$$

(ii) L.S. sol'n of $Ax = b$, $b = \begin{pmatrix}1\\2\\7\end{pmatrix}$

$$b^T q_1 = (1\ 2\ 7)\begin{pmatrix}1/3\\2/3\\-2/3\end{pmatrix} = \frac{1}{3} + \frac{4}{3} - \frac{14}{3} = -3$$

$$b^T q_2 = (1\ 2\ 7)\begin{pmatrix}2/3\\1/3\\2/3\end{pmatrix} = \frac{2}{3} + \frac{2}{3} + \frac{14}{3} = 6$$

$$\therefore \text{proj}_{C(A)} b = -3q_1 + 6q_2 = \begin{pmatrix}-1\\-2\\2\end{pmatrix} + \begin{pmatrix}4\\2\\4\end{pmatrix} = \begin{pmatrix}3\\0\\6\end{pmatrix} \equiv \hat{b}.$$

Solve $A\hat{x} = \hat{b} \rightarrow \boxed{\hat{x} = \begin{pmatrix}1\\2\end{pmatrix}}$

$\searrow$ least-squares sol'n

3.4

(16) $a_1 = \begin{pmatrix}1\\2\\2\end{pmatrix}$ $a_2 = \begin{pmatrix}1\\3\\1\end{pmatrix}$

$\|a_1\| = \sqrt{1^2+2^2+2^2} = 3 \Rightarrow \boxed{q_1 = \begin{pmatrix}1/3\\2/3\\2/3\end{pmatrix}}$ $a_2^T q_1 = (1\ 3\ 1)\begin{pmatrix}1/3\\2/3\\2/3\end{pmatrix} = \frac{1+6+2}{3} = 3$

$u_2 = a_2 - (a_2^T q_1)q_1 = a_2 - 3q_1 = \begin{pmatrix}1\\3\\1\end{pmatrix} - \begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$

$\|u_2\| = \sqrt{1^2+(-1)^2} = \sqrt{2} \Rightarrow \boxed{q_2 = \begin{pmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}}$

$$\text{so } A = \underset{3\times2}{\begin{pmatrix}1/3 & 0\\2/3 & 1/\sqrt{2}\\2/3 & -1/\sqrt{2}\end{pmatrix}}\underset{2\times2}{\begin{pmatrix}q_1^Ta_1 & q_1^Ta_2\\ & q_2^Ta_2\end{pmatrix}} = \boxed{\begin{pmatrix}1/3 & 0\\2/3 & 1/\sqrt{2}\\2/3 & -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}3 & 3\\0 & 2/\sqrt{2}\end{pmatrix} = QR}$$

$q_1^Ta_1 = 1/3 + 4/3 + 4/3$ $q_2^Ta_2 = (3-1)/\sqrt{2}$

$q_1^Ta_2 = 1/3 + 6/3 + 2/3$

Given n vectors a_i with m components,

A is m×n
Q is m×n
R is n×n

(21) $f(x) = \sin 2x \Rightarrow a = \text{proj}_{\cos(x)} \sin(2x) = \int_{-\pi}^{\pi} \sin(2x)\cos x\, dx =$

$b = \text{proj}_{\sin(x)} \sin(2x) = \int_{-\pi}^{\pi} \sin(2x)\sin x\, dx =$

(ii) Note $(1,x) = 0$ on $(-\pi,\pi) \rightarrow$ symmetric interval.

$$\text{so } c = \frac{(\sin 2x, 1)}{(1,1)} = \frac{\int_{-\pi}^{\pi} \sin 2x\, dx}{\int_{-\pi}^{\pi} 1\, dx} = 0$$

$$d = \frac{(\sin 2x, x)}{(x,x)} = \frac{\int_{-\pi}^{\pi} x\sin(2x)\, dx}{\int_{-\pi}^{\pi} x^2\, dx} = \ldots$$

3.4

(23) $y(x) = \begin{cases} 1 & 0 \le x \le \pi \\ 0 & \pi < x < 2\pi \end{cases}$

$$a_0 = \frac{(y,1)}{(1,1)} = \frac{\int_0^{\pi} 1\,dx}{\int_0^{2\pi} 1\,dx} = \frac{x|_0^{\pi}}{x|_0^{2\pi}} = \frac{\pi}{2\pi} = \boxed{\frac{1}{2} = a_0}$$

$$a_1 = \frac{(y,\cos x)}{(\cos x,\cos x)} = \frac{\int_0^{\pi} \cos x\,dx}{\int_0^{2\pi} \cos^2 x\,dx} = \frac{\sin x|_0^{\pi}}{\quad} = \boxed{0 = a_1}$$

$$b_1 = \frac{(y,\sin x)}{(\sin x,\sin x)} = \frac{\int_0^{\pi} \sin x\,dx}{\int_0^{2\pi} \sin^2 x\,dx} = \frac{-\cos x|_0^{\pi}}{\pi} = \frac{1+1}{\pi} = \boxed{\frac{2}{\pi} = b_1}$$

(24) We must orthogonalize $\underbrace{x^3+ax^2+bx+c}_{p}$:

The fist 3 legendre polys are:

$v_1 = 1$

$v_2 = x$

$v_3 = x^2 - 1/3$

So $$\boxed{v_4 = p - \frac{(p,v_1)}{(v_1,v_1)}1 - \frac{(p,v_2)}{(v_2,v_2)}x - \frac{(p,v_3)}{(v_3,v_3)}(x^2 - 1/3)} \quad (*)$$

Now compute the inner products:

- $(p,v_1) = \int_{-1}^{1} x^3+ax^2+bx+c = \frac{x^4}{4}\Big|_{-1}^{1} + \frac{ax^3}{3}\Big|_{-1}^{1} + \frac{bx^2}{2}\Big|_{-1}^{1} + cx\Big|_{-1}^{1}$

  $= 0 + a\left(\frac{1}{3} - -\frac{1}{3}\right) + 0 + c - -c$

  $= \boxed{\frac{2}{3}a + 2c}$

- $(v_1,v_1) = \int_{-1}^{1} 1\,dx = x\Big|_{-1}^{1} = 1-(-1) = \boxed{2}$
- $(p,v_2) = \int_{-1}^{1} x^4+ax^3+bx^2+cx = \frac{x^5}{5}\Big|_{-1}^{1} + \frac{bx^3}{3}\Big|_{-1}^{1} = \left(\frac{1}{5}+\frac{1}{5}\right)+\left(\frac{b}{3}+\frac{b}{3}\right)$

  $= \boxed{\frac{2}{5} + \frac{2b}{3}}$

- $(v_2,v_2) = \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - \left(\frac{-1}{3}\right) = \boxed{\frac{2}{3}}$
- $(p,v_3) = \ldots$
- $(v_3,v_3) = \ldots$

Now just plug inner products back in to (*).

$\boxed{3.5}$

⑪ $c = \begin{pmatrix}1\\0\\1\\0\end{pmatrix} \xrightarrow[\substack{\text{break}\\\text{into}\\\text{odd/even}}]{} \begin{pmatrix}1\\1\\0\\0\end{pmatrix} \xrightarrow[\substack{\text{compute}\\\text{smaller}\\\text{Fourier transforms.}}]{} \begin{pmatrix}2\\0\\2\\0\end{pmatrix}$. Details:

$$F_4 c = F_4 P_4 P_4^T c \quad \text{where } P_4 = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix} \Rightarrow P_4^T = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

$$F_4 = \begin{pmatrix}1&1&1&1\\1&w_4&w_4^2&w_4^3\\1&w_4^2&w_4^4&w_4^6\\1&w_4^3&w_4^6&w_4^9\end{pmatrix} \Rightarrow F_4 P_4 = \left(\begin{array}{cc|cc}1&1&1&1\\1&w_4^2&w_4&w_4^3\\\hline 1&w_4^4&w_4^2&w_4^6\\1&w_4^6&w_4^3&w_4^9\end{array}\right) = \left(\begin{array}{c|c}F_2&D_2F_2\\\hline F_2&-D_2F_2\end{array}\right)$$

$$\therefore\ F_4 c = \underbrace{\left(\begin{array}{c|c}F_2&D_2F_2\\\hline F_2&-D_2F_2\end{array}\right)}_{F_4P_4}\underbrace{\begin{pmatrix}1\\1\\\hline 0\\0\end{pmatrix}}_{P_4^Tc} = \begin{pmatrix}F_2\begin{pmatrix}1\\1\end{pmatrix}\\\hline F_2\begin{pmatrix}1\\1\end{pmatrix}\end{pmatrix}$$

$$F_2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1&1\\1&w_2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\1+w_2\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$

$$\Rightarrow \boxed{F_4 c = \begin{pmatrix}2\\0\\2\\0\end{pmatrix}}$$

$\boxed{A.2}$

② $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det A = \boxed{-1/8}$

$\det(-A) = (-1)^3 \det A = (-1)\cdot(-1) = \boxed{1}$

$\det(A^2) = \det(A \cdot A) = \det A \cdot \det A = \boxed{1}$

$\det(A^{-1}) = 1/\det A = 1/-1 = \boxed{-1}$

⑦ ⓐ $A = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} (2\ {-1}\ 2) = \left(\begin{pmatrix} 2 & -1 & 2 \\ 8 & -4 & 8 \\ 4 & -2 & 4 \end{pmatrix} \right)$

$\det A = 2\begin{vmatrix} -4 & 8 \\ -2 & 4 \end{vmatrix} + 1\begin{vmatrix} 8 & 8 \\ 4 & 4 \end{vmatrix} + 2\begin{vmatrix} 8 & -4 \\ 4 & -2 \end{vmatrix}$

$= 2(-16+16) + 1(32-32) + 2(-16+16) = \boxed{0}$

which makes sense since A is rank 1 and ∴ singular!

ⓑ $\det U = 4(1)(2)(2) = \boxed{16}$

ⓒ $\det(U^T) = \det U = \boxed{16}$

ⓓ $\det(U^{-1}) = 1/\det U = \boxed{1/16}$

ⓔ 2 row interchanges ($R1 \Leftrightarrow R4$ and $R2 \Leftrightarrow R3$)

$\Rightarrow \det(M) = (-1)(-1)\det U = \det U = \boxed{16}$

⑩ $Q^TQ = I \Rightarrow \det(Q^TQ) = \det(I)$

$\Rightarrow \det(Q^T)\det(Q) = \det(I) = 1$

$\Rightarrow (\det Q)^2 = 1$ since $\det Q^T = \det Q$

$\Rightarrow \det Q = \pm 1$ ☑

The rows (and cols) of Q form a $\boxed{\text{unit cube.}}$

4.2

⑫

$$\begin{vmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{vmatrix} = \begin{vmatrix}1&1&1\\a&b&c\\a^2&b^2&c^2\end{vmatrix} = \begin{vmatrix}1&1&1\\0&b-a&c-a\\0&b^2-a^2&c^2-a^2\end{vmatrix}$$

$$= \begin{vmatrix}1&1&1\\0&b-a&c-a\\0&0&(c^2-a^2)-(b+a)(c-a)\end{vmatrix}$$

$$= 1(b-a)[(c^2-a^2)-(b+a)(c-a)]$$

$$= 1(b-a)[(\underline{c-a})(c+a)-(b+a)(\underline{c-a})]$$

$$= (b-a)(c-a)[(c+a)-(b+a)]$$

$$= (b-a)(c-a)(c-b)$$

☑

⑭ ⓐ $\boxed{\text{False}}$ $A=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ $B=\begin{pmatrix}2&1\\1&1\end{pmatrix}$

$\Rightarrow$ det $A=0$, det $B=2-1=1\neq 2(0)$.

ⓑ True, up to a sign change - see p. 205

ⓒ $\boxed{\text{False}}$ $A=\begin{pmatrix}-1&1\\1&1\end{pmatrix}$ invertible, $B=\begin{pmatrix}2&0\\0&0\end{pmatrix}$ singular $\Big\}$ $A+B=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ not inv'ble!

ⓓ $\boxed{\text{True}}$ det(AB) = det A det B

= det $A \cdot 0$ since B singular

$=0$

∴ AB is singular

ⓔ $\boxed{\text{False}}$ $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ $B=\begin{pmatrix}2&1\\-1&3\end{pmatrix}$

$AB=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}2&1\\-1&3\end{pmatrix}=\begin{pmatrix}0&7\\2&15\end{pmatrix}$, $BA=\begin{pmatrix}2&1\\-1&3\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}5&8\\8&10\end{pmatrix}$

$|AB-BA|=\begin{vmatrix}-5&-1\\-6&5\end{vmatrix}\neq 0$

⑱ $\boxed{4.2}$

$\det A = -4$ $\quad$ $\det AB = \det A \cdot \det B = -16$

$\det B = 4$ $\quad$ $\det A^T A = \det A^T \det A = (\det A)^2 = 16$

$\det C = 0$ $\quad$ $\det C^T = \det C = 0$

㉕ $\left.\begin{array}{l} \det L = 1\cdot 1\cdot 1 = \boxed{1} \\ \det U = 3\cdot 2\cdot(-1) = \boxed{-6} \end{array}\right\} \Rightarrow \det A = \det L \cdot \det U = \boxed{-6}$

$\det(U^{-1}L^{-1}) = \det((LU)^{-1}) = 1/\det LU = \boxed{-1/6}$

$\det(U^{-1}L^{-1}A) = \det(U^{-1}L^{-1}LU) = \det(I) = \boxed{1}$

㉘ $|A| = \begin{vmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{vmatrix} = -\begin{vmatrix} c & 0 & 0 \\ 0 & 0 & b \\ 0 & a & 0 \end{vmatrix} = -\left(-\begin{vmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{vmatrix}\right) = \boxed{abc}$

$|B| = (-1)^3 \begin{vmatrix} d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{vmatrix} = \boxed{-abcd}$

$|C| = \begin{vmatrix} a & a & a \\ a & b & b \\ a & b & c \end{vmatrix} = \begin{vmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & b-a & c-a \end{vmatrix} = \begin{vmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & 0 & (c-a)-(b-a) \end{vmatrix}$

$= [a(b-a)][(c-a)-(b-a)]$

$= \boxed{a(b-a)(c-b)}$

㉛ $\det(\text{hilb}(1)) = 1$

$\det(\text{hilb}(2)) = .0833$

" " $3 = 4.7 \times 10^{-4}$

" " $4 = 1.7 \times 10^{-7}$

" " $5 = 3.8 \times 10^{-12}$

" " $6 = 5.4 \times 10^{-18}$

" " $7 = 4.8 \times 10^{-25}$

" " $8 = 2.7 \times 10^{-33}$

" " $9 = 9.7 \times 10^{-43}$

" " $10 = 2.16 \times 10^{-53}$