Math 102
Winter 2008
Homework #5

$\boxed{3.1}$

(2) $\left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\5\end{pmatrix} \right\}$ is a $\underbrace{\text{L.I. set}}$ that is $\underbrace{\text{not orthogonal}}$

$c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ only if $c_1 = c_2 = 0$

$(1\ 1)\begin{pmatrix}2\\5\end{pmatrix} = 1\cdot 2 + 1\cdot 5 = 7 \neq 0.$

$\left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix} \right\}$ is an $\underbrace{\text{orthogonal set}}$ that is $\underbrace{\text{not L.I.}}$

$(1\ 1)\begin{pmatrix}0\\0\end{pmatrix} = 0+0=0$

$0\begin{pmatrix}1\\1\end{pmatrix} + c\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ for all $c \in \mathbb{R}$!

(6) $(x\ y\ z)\begin{pmatrix}1\\1\\1\end{pmatrix} = 0 \Rightarrow x+y+z=0$

$(x\ y\ z)\begin{pmatrix}1\\-1\\0\end{pmatrix} = 0 \Rightarrow x-y+0=0$

This is what it means for $\begin{pmatrix}x\\y\\z\end{pmatrix}$ to be orthogonal to both $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\0\end{pmatrix}$.

So all vectors orthog to both ↗ and ↗ must satisfy

$\begin{matrix} x+y+z=0 \\ x=y \end{matrix} \Rightarrow \begin{matrix} z=-x-y=-2y \\ x=y \end{matrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}y\\y\\-2y\end{pmatrix} = y\begin{pmatrix}1\\1\\-2\end{pmatrix}$

So the subspace in question has basis $\left\{ \begin{pmatrix}1\\1\\-2\end{pmatrix} \right\}$

It only has 1 vector, so it's trivially an orthogonal basis; to make it orthonormal, we need only normalize the vector:

$$\left\| \begin{pmatrix}1\\1\\-2\end{pmatrix} \right\| = \sqrt{1^2+1^2+(-2)^2} = \sqrt{6}$$

$\therefore$ an O.N.B. for the subspace is $\boxed{\left\{ \begin{pmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{pmatrix} \right\}}$.

(7) Note: $\text{Row } A = (\text{Nul } A)^\perp$ $\quad A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{pmatrix}$.

$\text{Col } A = (\text{Nul } A^T)^\perp$

(i) $x \in (\text{Row } A)^\perp \Rightarrow x \in \text{Nul } A$. Compute Nul A: $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 4 & 3 & 0 \\ 3 & 6 & 4 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 + 2x_2 + x_3 = 0 \\ x_3 = 0 \end{array}$ $\quad \therefore x = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \in \text{Nul } A \Rightarrow \boxed{\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \in (\text{Row } A)^\perp}$

(ii) $y \in (\text{Col } A)^\perp \Rightarrow y \in \text{Nul } A^T$. Compute $\text{Nul}(A^T)$:

$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \Rightarrow \begin{array}{l} y_1 + 2y_2 + 3y_3 = 0 \\ y_2 + y_3 = 0 \end{array}$ $\quad \therefore \boxed{y = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \in (\text{Col } A)^\perp}$

(iii) $z \in (\text{Nul } A)^\perp \Rightarrow z \in \text{Row } A$. Take $\boxed{z = (1\ 2\ 1)}$

Note THESE ANSWERS ARE NOT UNIQUE!

(11) Claim: For $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, one and only one of the following systems has a solution:

(i) $Ax = b$

(ii) $A^T y = 0$ for some y s.t. $y^T b \neq 0$.

Pf: Assume, for a contradiction, that both (i) and (ii) have solutions. Then

$$0 \neq y^T b = y^T(Ax) = [y^T(Ax)]^T = (Ax)^T y = x^T \underbrace{A^T y}_{0} = 0$$

$\therefore 0 \neq 0$

This is absurd. Thus, it is not possible for both (i) and (ii) to have solutions ✓

It remains to show that either (i) or (ii) must hold. Assume (i) has no solution. Then

$b \notin \text{Col } A$

$\Rightarrow b \notin (\text{Nul } A^T)^\perp$

$\Rightarrow \exists\, y \in \text{Nul } A^T$ s.t. $y^T b \neq 0$

which precisely says that (ii) has a solution.

$\therefore$ Either (i) or (ii) has a solution.

Conclude that exactly one of (i), (ii) has a solution ☑

(14) Claim: $x-y$ orthogonal to $x+y$ $\Leftrightarrow$ $\|x\|=\|y\|$

Pf. ($\Rightarrow$) $x-y$ orthogonal to $x+y$

$\Rightarrow (x-y)^T(x+y)=0$

$\Rightarrow (x^T-y^T)(x+y)=0$

$\Rightarrow x^Tx - y^Tx + x^Ty - y^Ty = 0$

But y^Tx is a <u>scalar</u>, so $y^Tx = (y^Tx)^T = x^Ty$

Also, $x^Tx = \|x\|^2 \geq 0$ and $y^Ty = \|y\|^2 \geq 0$

$\therefore \|x\|^2 - \cancel{y^Tx + y^Tx} - \|y\|^2 = 0 \Rightarrow \|x\|^2 = \|y\|^2$

$\Rightarrow \|x\| = \|y\|$ ✓

($\Leftarrow$) Assume $\|x\|=\|y\|$

<u>WTS</u>: $(x-y)^T(x+y)=0$

$(x-y)^T(x+y) = (x^T-y^T)(x+y)$ >0 (see above)

$= x^Tx - \cancel{y^Tx + x^Ty} - y^Ty$

$= x^Tx - y^Ty$

$= \|x\|^2 - \|y\|^2$

$=0$ by assumption ✓

$\boxed{\checkmark}$

(16) Let $S = \left\{ x \in \mathbb{R}^4 \mid x \text{ is orthogonal to both } \begin{pmatrix}1\\4\\4\\1\end{pmatrix} \text{ and } \begin{pmatrix}2\\9\\8\\2\end{pmatrix} \right\}$

Then $(x_1\ x_2\ x_3\ x_4)\begin{pmatrix}1\\4\\4\\1\end{pmatrix} = 0$ <u>and</u> $(x_1\ x_2\ x_3\ x_4)\begin{pmatrix}2\\9\\8\\2\end{pmatrix} = 0$

by definition of orthogonality, which gives us the system of equations : $x_1 + 4x_2 + 4x_3 + x_4 = 0$

$2x_1 + 9x_2 + 8x_3 + 2x_4 = 0$

$\Rightarrow \left(\begin{array}{cccc|c} 1 & 4 & 4 & 1 & 0 \\ 2 & 9 & 8 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 4 & 4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 + 4x_2 + 4x_3 + x_4 = 0 \\ x_2 = 0 \end{array}$

$\Rightarrow x_1 = -4x_3 - x_4$

So we can completely describe S!

$$S = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}-4x_3-x_4\\0\\x_3\\x_4\end{pmatrix} \right\} = \boxed{\left\{ x_3\begin{pmatrix}-4\\0\\1\\0\end{pmatrix} + x_4\begin{pmatrix}-1\\0\\0\\1\end{pmatrix} \text{ s.t. } x_3, x_4 \in \mathbb{R} \right\}}$$

(19) (A) If V, W are lines in $\mathbb{R}^3$ then $V^\perp, W^\perp$ are planes in $\mathbb{R}^3$. Two planes in $\mathbb{R}^3$ can't be orthogonal since they are either parallel or intersect in a line.

(B) Let $V = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$

$W = \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$

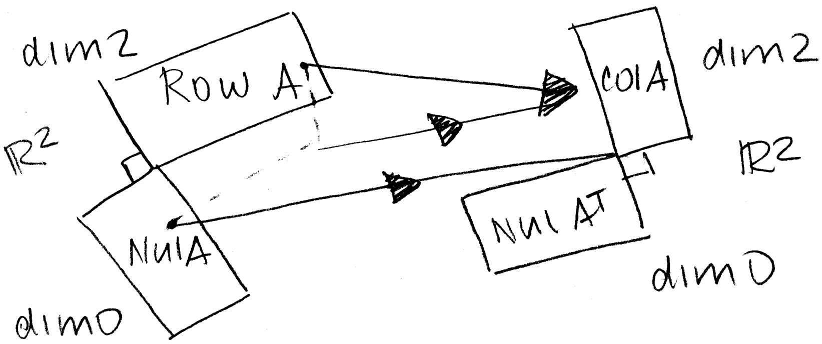
$Z = \text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$

You check that $V \perp W$, $W \perp Z$, but V and Z are <u>not</u> orthogonal!

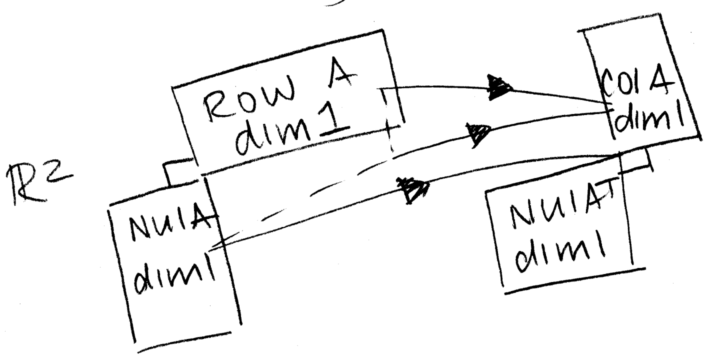
(22) Let $A = (1\ 1\ 1\ 1)$. Then $S = \text{Nul}\, A$.

$\Rightarrow S^\perp = (\text{Nul}\, A)^\perp = \text{Row}\, A$, $\boxed{\text{which has basis } \{(1\ 1\ 1\ 1)\}}$

(32) $A = \begin{pmatrix}1 & 2\\3 & 6\end{pmatrix}$



$B = \begin{pmatrix}1 & 0\\3 & 0\end{pmatrix}$



(37) Claim: Every $y \in N(A^T)$ is perpendicular to every $Ax \in \text{Col}(A)$.

Pf: Assume $y \in N(A^T)$. Then $A^T y = 0$.

Assume $Ax \in \text{Col}(A)$.

Then $(Ax)^T y = (x^T A^T) y = x^T (A^T y) = x^T 0 = 0$.

$\therefore$ Ax and y are orthogonal $\boxed{\checkmark}$

(44) Claim: If a subspace S is contained in a subspace V, then $S^\perp$ contains $V^\perp$.

Pf: Assume $S \subset V$.

WTS: $V^\perp \subset S^\perp$.

Let $x \in V^\perp$. Then $v^T x = 0 \quad \forall\, v \in V$. $\quad (*)$

Let $s \in S$. We must show $s^T x = 0$.

But $S \subset V$, so $s \in S \Rightarrow s \in V$.

$\therefore$ $s^T x = 0$, by $(*)$.

$\Rightarrow x \in S^\perp$ $\boxed{\checkmark}$

(51) $\text{Rank}(B) + \dim(\text{Nul } B) = 5$

Let $Bx \in \text{Col} B$. Then $A(Bx) = (AB)x = 0x = 0$

$\Rightarrow \text{Col } B \subset \text{Nul } A$

$\Rightarrow \text{rank } B \leq \dim(\text{Nul } A) = 4 - \text{rank}(A)$.

since A is 3×4

$\Rightarrow \text{rank } B + \text{rank } A \leq 4$. $\boxed{\checkmark}$

3.2

(14) I. Find intersection of the planes $x+y+t=0$ and $x-t=0$.

Notice that this intersection is just the solutions to the system: $\begin{matrix} x+y+t=0 \\ x-t=0 \end{matrix} \Rightarrow \begin{matrix} x+y+t=0 \\ x=t \end{matrix} \Rightarrow 2t=-y \Rightarrow y=-2t$

$\therefore$ The intersection $= \left\{ \begin{pmatrix} x \\ y \\ t \end{pmatrix} = \begin{pmatrix} t \\ -2t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\} = sp\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$.

II. Find matrix P that projects onto the line thru $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

By Formula (17), $P = \dfrac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}(1\ -2\ 1)}{(1\ -2\ 1)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$

$= \boxed{\frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}}$

(17) (A) $\text{Proj}_a b = \dfrac{(1\ 2\ 2)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{(1\ 1\ 1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{5}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 5/3 \\ 5/3 \end{pmatrix} = P$

$e = b - p = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 5/3 \\ 5/3 \\ 5/3 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$

$e^T a = (-2/3\ \ 1/3\ \ 1/3)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -2/3 + 1/3 + 1/3 = 0$ $\Rightarrow$ e is perpendicular to a ✓

(B) same computation as (A)

(19) (A) $P = \frac{aa^T}{a^Ta} = \frac{\begin{pmatrix}1\\1\\1\end{pmatrix}(1\,1\,1)}{(1\,1\,1)\begin{pmatrix}1\\1\\1\end{pmatrix}} = \frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} = \begin{pmatrix}1/3&1/3&1/3\\1/3&1/3&1/3\\1/3&1/3&1/3\end{pmatrix}$.

$P^2 = \begin{pmatrix}1/3&1/3&1/3\\1/3&1/3&1/3\\1/3&1/3&1/3\end{pmatrix}\begin{pmatrix}1/3&1/3&1/3\\1/3&1/3&1/3\\1/3&1/3&1/3\end{pmatrix} = \begin{pmatrix}1/3&1/3&1/3\\1/3&1/3&1/3\\1/3&1/3&1/3\end{pmatrix} = P$ ✓

$Pb = \begin{pmatrix}1/3&1/3&1/3\\1/3&1/3&1/3\\1/3&1/3&1/3\end{pmatrix}\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}5/3\\5/3\\5/3\end{pmatrix}$ which agrees with what we got in (17). Linear algebra works!

(B) Similar computation.

(21) $a_1 = \begin{pmatrix}-1\\2\\2\end{pmatrix}$ $a_2 = \begin{pmatrix}2\\2\\-1\end{pmatrix}$.

$P_1 = \frac{a_1a_1^T}{a_1^Ta_1} = \frac{\begin{pmatrix}-1\\2\\2\end{pmatrix}(-1\,2\,2)}{(-1\,2\,2)\begin{pmatrix}-1\\2\\2\end{pmatrix}} = \frac{1}{9}\begin{pmatrix}1&-2&-2\\-2&4&4\\-2&4&4\end{pmatrix}$

$P_2 = \frac{a_2a_2^T}{a_2^Ta_2} = \frac{\begin{pmatrix}2\\2\\-1\end{pmatrix}(2\,2\,-1)}{(2\,2\,-1)\begin{pmatrix}2\\2\\-1\end{pmatrix}} = \frac{1}{9}\begin{pmatrix}4&4&-2\\4&4&-2\\-2&-2&1\end{pmatrix}$

$P_1P_2 = \frac{1}{81}\begin{pmatrix}1&-2&-2\\-2&4&4\\-2&4&4\end{pmatrix}\begin{pmatrix}4&4&-2\\4&4&-2\\-2&-2&1\end{pmatrix} = \frac{1}{81}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

Why does this make sense? P_1P_2 represents the linear transformation that first projects a vector onto the line thru a_2, and then projects that projected vector onto the line thru a_1.
<u>But</u> a_1 is perpendicular to a_2 (check: $a_1^Ta_2 = 0$) so the resulting projection leaves us with <u>nothing</u>

$\boxed{3.3}$

④ (i) $\underbrace{\|Ax-b\|^2}_{E(x)} = (Ax-b)^T(Ax-b)$

$$= (x^TA^T - b^T)(Ax-b)$$
$$= x^TA^TAx - x^TA^Tb - b^TAx + b^Tb.$$

$\underline{x^TA^TAx} = (u\ v)\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}$

$$= (u\ v)\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} = (u\ v)\begin{pmatrix}2u+v\\u+2v\end{pmatrix} = u(2u+v)+v(u+2v)$$
$$=2u^2+2uv+2v^2.$$

$\underline{x^TA^Tb} = (u\ v)\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}\begin{pmatrix}1\\3\\4\end{pmatrix} = (u\ v)\begin{pmatrix}5\\7\end{pmatrix} = 5u+7v$

$\underline{b^TAx} = (x^TA^Tb)^T \underset{\uparrow}{=} x^TA^Tb = 5u+7v$

since $x^TA^Tb \in \mathbb{R}$.

$\underline{b^Tb} = (1\ 3\ 4)\begin{pmatrix}1\\3\\4\end{pmatrix} = 1+9+16 = 26.$

$\therefore\ E(x(u,v)) = 2u^2+2uv+2v^2-5u-7v-5u-7v+26$
$$=2u^2+2v^2+2uv-10u-14v+26$$

$\Rightarrow \dfrac{\partial E}{\partial u} = 4u+2v-10, \quad \dfrac{\partial E}{\partial v} = 4v+2u-14$

Setting the partial derivatives $=0$ gives: $\boxed{\begin{aligned}4u+2v-10=0\\4v+2u-14=0\end{aligned}}$

(ii) consider the normal eqns: $A^TAx = A^Tb$:

$$\underset{A^TA}{\begin{pmatrix}2&1\\1&2\end{pmatrix}}\begin{pmatrix}u\\v\end{pmatrix} = \underset{A^Tb}{\begin{pmatrix}5\\7\end{pmatrix}} \Rightarrow \begin{pmatrix}2u+v\\u+2v\end{pmatrix} = \begin{pmatrix}5\\7\end{pmatrix}$$

$$\Rightarrow \boxed{\begin{aligned}2u+v=5\\u+2v=7\end{aligned}}$$

A tiny bit of algebra shows that these are really the same systems! So, calculus does agree with linear algebra.

4, cont'd

(iii) Find solution $\hat{x}$:

To do this we solve $A^TAx = A^Tb$:

$$\left(\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & 2 & 7 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & 1 & 5 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & -3 & -9 \end{array}\right) \Rightarrow \begin{array}{l} u+2v=7 \\ -3v=-9 \end{array} \Rightarrow \begin{array}{l} u=1 \\ v=3 \end{array}$$

$$\Rightarrow \boxed{\hat{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

(iv) $$p = A\hat{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = b$$

$p = b$ since $b \in \text{Col } A$ to begin with!

⑥ (i) $Pb = A(A^TA)^{-1}A^Tb$ (equation 4).

$$A^TA = \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -8 \\ -8 & 18 \end{pmatrix} \Rightarrow (A^TA)^{-1} = \frac{1}{18(6)-64}\begin{pmatrix} 18 & 8 \\ 8 & 6 \end{pmatrix}$$

$$= \frac{1}{44}\begin{pmatrix} 18 & 8 \\ 8 & 6 \end{pmatrix}$$

$$\therefore \underline{\boxed{Pb}} = \frac{1}{44}\underset{3\times 2}{\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix}}\underset{2\times 2}{\begin{pmatrix} 18 & 8 \\ 8 & 6 \end{pmatrix}}\underset{2\times 3}{\begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{pmatrix}}\underset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}}$$

$$= \boxed{\begin{pmatrix} 2.0909 \\ -1.2727 \\ 5.9091 \end{pmatrix}}$$

(Feel free to use Matlab to do these types of computations).

(ii) $$p = \begin{pmatrix} 2.0909 \\ -1.2727 \\ 5.9091 \end{pmatrix} \in \text{Col } A$$

$$q = b - p = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} - \begin{pmatrix} 2.0909 \\ -1.2727 \\ 5.9091 \end{pmatrix} = \begin{pmatrix} -1.0909 \\ 3.2727 \\ 1.0909 \end{pmatrix}$$

<u>check</u> $q^Tp = 0$ (so q and p are orthogonal).

(iii) $q \in (\text{Col } A)^{\perp} = \text{Nul}(A^T)$.

(12) $V = \operatorname{sp}\left\{ \begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right\}$

(a) Let $A = \begin{pmatrix}1&0\\1&0\\0&1\\1&0\end{pmatrix}$. Then $V = \operatorname{Col}(A)$

$[\operatorname{Col}(A)]^{\perp} = \operatorname{Nul} A^T$, so find basis for $\operatorname{Nul}(A^T)$:

$$\left(\begin{array}{cccc|c}1&1&0&1&0\\0&0&1&0&0\end{array}\right) \Rightarrow \begin{array}{l} x_1+x_2+x_4=0 \Rightarrow x_1=-x_2-x_4 \\ x_3=0 \end{array}$$

$$\Rightarrow \operatorname{Nul}(A^T) = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}-x_2-x_4\\x_2\\0\\x_4\end{pmatrix} = x_2\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}-1\\0\\0\\1\end{pmatrix} \right\}.$$

$\therefore$ $\boxed{\left\{ \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix} \right\} \text{ is a basis for } V^{\perp}}$

(b) Projection onto V = Projection onto $\operatorname{Col}(A)$, so

$$P = A(A^TA)^{-1}A^T$$

$$A^TA = \begin{pmatrix}1&1&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1&0\\1&0\\0&1\\1&0\end{pmatrix} = \begin{pmatrix}3&0\\0&1\end{pmatrix} \Rightarrow (A^TA)^{-1} = \frac{1}{3}\begin{pmatrix}1&0\\0&3\end{pmatrix}$$

$$A(A^TA)^{-1} = \frac{1}{3}\begin{pmatrix}1&0\\1&0\\0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&3\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1&0\\1&0\\0&3\\1&0\end{pmatrix}$$

$$A(A^TA)^{-1}A^T = \frac{1}{3}\begin{pmatrix}1&0\\1&0\\0&3\\1&0\end{pmatrix}\begin{pmatrix}1&1&0&1\\0&0&1&0\end{pmatrix} = \boxed{\frac{1}{3}\begin{pmatrix}1&1&0&1\\1&1&0&1\\0&0&3&0\\1&1&0&1\end{pmatrix} = P}$$

(c) Just project b onto V using the P we found!

$$\operatorname{Proj}_V b = \frac{1}{3}\begin{pmatrix}1&1&0&1\\1&1&0&1\\0&0&3&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \boxed{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}}$$

(14) Plane $= sp\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$.

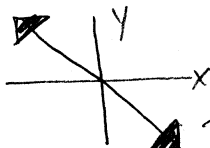
Let $A=\begin{pmatrix}1&1\\1&1\\0&1\end{pmatrix}$ Then Plane $= col(A)$.

$P=$ projection matrix onto $col(A) = A(A^TA)^{-1}A^T$

$$=\begin{pmatrix}1/2&1/2&0\\1/2&1/2&0\\0&0&1\end{pmatrix}$$

To find a $b\neq 0$ that is projected to $\hat{0}$, find $b\in Nul P$!
(and check that $Pb=0$!)

(17) $x+y=0 \Rightarrow x=-y$
$\Rightarrow y=-x$



line thru $\begin{pmatrix}-1\\1\end{pmatrix}$

$\mathcal{E}=\{e_1,e_2\}=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ = std basis for $\mathbb{R}^2$.

Let $A=$ matrix that projects $\mathbb{R}^2$ onto above line.

Let T represent the projection ("T" for transformation)

Then $A=\left(T(e_1)\,\middle|\,T(e_2)\right)$.

$$T(e_1)=\frac{e_1^T\begin{pmatrix}-1\\1\end{pmatrix}}{(-1\ 1)\begin{pmatrix}-1\\1\end{pmatrix}}\begin{pmatrix}-1\\1\end{pmatrix}=\frac{(1\ 0)\begin{pmatrix}-1\\1\end{pmatrix}}{(-1\ 1)\begin{pmatrix}-1\\1\end{pmatrix}}\begin{pmatrix}-1\\1\end{pmatrix}=\frac{-1}{2}\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}1/2\\-1/2\end{pmatrix}$$

$$T(e_2)=\frac{e_2^T\begin{pmatrix}-1\\1\end{pmatrix}}{(-1\ 1)\begin{pmatrix}-1\\1\end{pmatrix}}\begin{pmatrix}-1\\1\end{pmatrix}=\frac{(0\ 1)\begin{pmatrix}-1\\1\end{pmatrix}}{(-1\ 1)\begin{pmatrix}-1\\1\end{pmatrix}}\begin{pmatrix}-1\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}-1/2\\1/2\end{pmatrix}.$$

$$\boxed{\therefore A=\begin{pmatrix}1/2&-1/2\\-1/2&1/2\end{pmatrix}}$$

(22) Set up linear system using data:

$$\begin{aligned} C-2D &= 4 \\ C-D &= 2 \\ C+0D &= -1 \\ C+D &= 0 \\ C+2D &= 0 \end{aligned} \longrightarrow \underbrace{\begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}}_{A} \begin{pmatrix} C \\ D \end{pmatrix} = \underbrace{\begin{pmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{b}$$

Find least-squares soln to $Ax=b$

Solve normal eqns: $A^TA\,x = A^Tb$.

$$\Rightarrow x = \underbrace{(A^TA)^{-1}}_{}A^Tb$$

A has full col. rank, so A^TA is invertible.

$\therefore \boxed{x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

(26) $L = a + bF$

$$\begin{aligned} 5 &= a + 1b \\ 6 &= a + 2b \\ 7 &= a + 4b \end{aligned} \longrightarrow \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}}_{c}$$

Find L.S. sol'n to $Ax = c \Rightarrow$ solve $A^TA\,x = A^Tc$

$\Rightarrow x = (A^TA)^{-1}A^Tc$ (again, A has full col rank so A^TA is invertible).

$$\Rightarrow x = \begin{pmatrix} 4.50 \\ 0.643 \end{pmatrix}$$

$\therefore \boxed{a = 4.5}$

(27)

(A) To find least squares soln to $ax = b$, we solve

$$a^T a \hat{x} = a^T b.$$

$$a^T a = \underbrace{(1 \cdots 1)}_{m} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad a^T b = (1 \cdots 1) \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = b_1 + b_2 + \cdots + b_m$$

$$\Rightarrow m \hat{x} = b_1 + b_2 + \cdots + b_m$$

$$\Rightarrow \hat{x} = \frac{b_1 + \cdots + b_m}{m} = \text{avg of } b_i\text{'s}$$

(B) $e = b - a\hat{x} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \dfrac{b_1 + \cdots + b_m}{m}$

$$= \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} - \begin{pmatrix} (b_1 + \cdots + b_m)/m \\ \vdots \\ (b_1 + \cdots + b_m)/m \end{pmatrix} = \begin{pmatrix} b_1 - \hat{x} \\ \vdots \\ b_m - \hat{x} \end{pmatrix}$$

$$\|e\| = \sqrt{(b_1 - \hat{x})^2 + \cdots + (b_m - \hat{x})^2}$$

$$\|e\|^2 = (b_1 - \hat{x})^2 + \cdots + (b_m - \hat{x})^2$$

(C) $e = b - a\hat{x} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} 3 = \begin{pmatrix} 1-3 \\ 2-3 \\ 6-3 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}.$

$$p^T e = (3\ 3\ 3) \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} = -6 - 3 + 9 = 0 \checkmark$$

Projection matrix $P = a(a^T a)^{-1} a^T \quad \left(a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)$

$$a^T a = (1\ 1\ 1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3 \Rightarrow (a^T a)^{-1} = 1/3$$

$$\therefore P = \frac{1}{3} a a^T = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (1\ 1\ 1) = \boxed{\frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = P}$$