

2.2

$$\textcircled{5} \textcircled{A} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 4 & 5 & 4 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

$$\Rightarrow u + 2v + 2w = 1 \Rightarrow u + 2v + 4 = 1 \Rightarrow u = -2v - 3$$

$$w = 2$$

$$\Rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2v - 3 \\ v \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} + v \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\textcircled{B} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 4 & 4 & 4 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array} \right) \Rightarrow \boxed{\text{no solution}}$$

$$\textcircled{8} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 2 & 4 & 0 & 7 & b_2 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - 2b_1 \end{array} \right)$$

consistent for all choices of b_1 and b_2 .

⑪ "x_N" just means "a vector in the nullspace of A"

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $Ax = b$ has no solution,

but A does have a nullspace:

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$$

$$\Rightarrow \text{Nul } A = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

So take $x_N = \begin{pmatrix} -12 \\ 12 \end{pmatrix}$, or any element of $\text{span} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

Claim: $\text{rank}(AB) \leq \text{rank}(B)$

Proof: Let A be $m \times n$, B be $n \times r \Rightarrow AB$ is $m \times r$

Recall the Rank + Nullity Thm:

$$\text{dim}(\text{Nul}(AB)) + \text{rk}(AB) = k$$

$$\text{dim}(\text{Nul}(B)) + \text{rk}(B) = k$$

(*)

Let $x \in \mathbb{R}^r$ be an element of $\text{Nul}(B)$

$$\Rightarrow Bx = 0 \quad (\text{defn of "nullspace"})$$

$$\Rightarrow ABx = 0$$

$$\Rightarrow x \in \text{Nul}(AB)$$

$$\text{Nul}(B) \subseteq \text{Nul}(AB)$$

$$\Rightarrow \text{dim}(\text{Nul}(B)) \leq \text{dim}(\text{Nul}(AB))$$

But then (*) implies that $\text{rk}(B) \geq \text{rk}(AB)$



A, B $n \times n$ such that $AB = I$

Claim: $BA = I$

Proof: $AB = I$, so $\text{rank}(AB) = n$ (since I has full rk.)

From (24), $\text{rank}(A) \geq \text{rank}(AB)$

$$\text{rank}(A) \geq n \quad (!)$$

But A is $n \times n$, so $\text{rank}(A) \leq n$ (!!)

$$(!) + (!!) \Rightarrow \text{rank}(A) = n$$

$\text{rank}(A)$ is invertible (square with full rank)

$$\Rightarrow B = A^{-1}$$

$$\Rightarrow BA = I$$



53) (A) False. Let $M = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$; $b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. Consider $Mx = b$.

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right) \Rightarrow x_1 + x_2 = 3 \Rightarrow x_1 = 3 - x_2$$

free variable

Key: Choose a square matrix with a nontrivial nullspace!

(B) True: $Ax = 0 \Rightarrow A^{-1}(Ax) = A^{-1}(0) \Rightarrow x = 0$.

So $\text{Nul } A = \{0\}$

∴ no free variables in any system $Ax = b$.

(C) True A $m \times n$

pivot variables $\leq \min\{m, n\} \leq n$.

(D) True: A $m \times n$

pivot variables $\leq \min\{m, n\} \leq m$

(70) Changing a matrix from echelon form to reduced echelon form makes the leading entry of each row equal to 1.

2.3

$\equiv A$

$$\begin{aligned}
 (2) \quad & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix} \\
 & \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

3 pivots $\Rightarrow \text{rk}(A) = 3$

$\Rightarrow \text{max \# of L.I. cols} = \boxed{3}$

(9) (A) ... $Ax = 0$ has a nonzero solution since A is 3×4 , and therefore there is at least one free variable in the homogeneous system.

(B) ... they are scalar multiples of each other ($v_1 = \alpha v_2$ for some $\alpha \in \mathbb{R}$)

(C) ... $0 \cdot v_1 + \alpha \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$ for all choices of $\alpha \in \mathbb{R}$
 \circ the homogeneous eqn. has a nontrivial solution
 \circ the vectors are linearly dependent.

(12) b is in $\text{span}\{\text{cols of } A\}$ when $Ax = b$ has a solution.

c is in $\text{span}\{\text{rows of } A\}$ when $A^T x = c$ has a solution.

False every vector space contains the zero vector, and $\text{Rowsp}(A)$ is always a vector space (regardless of independence of rows).

$$\textcircled{B} \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{col } A: \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\uparrow \quad \uparrow$
 pivot columns

$$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right\} \text{ is a basis for col } A$$

$$\Rightarrow \dim(\text{col } A) = 2.$$

col U: has same RREF as A and so same # of pivot cols, but

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\} \text{ is a basis for col } U.$$

$$\Rightarrow \dim(\text{col } A) = 2 = \dim(\text{col } U).$$

Row A: Row A = col(A^T)

$$A^T = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\uparrow \quad \uparrow$
 pivot cols

$$\therefore \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right\} \text{ is basis for col}(A^T) = \text{Row}(A)$$

$$\Rightarrow \dim(\text{Row } A) = 2.$$

Row U: Row U = col(U^T).

$$U^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\uparrow \quad \uparrow$
 pivot cols

$$\therefore \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\} \text{ is basis for col}(U^T) = \text{Row}(U).$$

$$\Rightarrow \dim(\text{Row } U) = 2.$$

Note: $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \therefore \text{Row } U = \text{Row } A.$

$$\textcircled{20} \textcircled{A} \quad \mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ such that } x=y=z=w \right\}$$

$$= \left\{ \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix} \text{ such that } x \in \mathbb{R} \right\}$$

$$= \left\{ x \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ such that } x \in \mathbb{R} \right\}$$

$\therefore \mathcal{S}$ has basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

$$\textcircled{B} \quad \mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ such that } x+y+z+w=0 \right\}$$

$$\downarrow \\ x = -y-z-w$$

$$= \left\{ \begin{pmatrix} -y-z-w \\ y \\ z \\ w \end{pmatrix} \text{ such that } y, z, w \in \mathbb{R} \right\}$$

$$= \left\{ y \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\therefore \mathcal{S}$ has basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$$\textcircled{C} \quad \mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ s.t. } (x \ y \ z \ w) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \ \& \ (x \ y \ z \ w) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = 0 \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ s.t. } x+y=0, x+z+w=0 \right\}$$

$$\Rightarrow y = z+w$$

x free

z, w free.



$$= \left\{ \begin{pmatrix} x \\ z+w \\ z \\ w \end{pmatrix} \right\} = \left\{ x \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

∴ \mathcal{S} has basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$$\textcircled{D} U = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

↑ ↑
PIVOT COLS

∴ Basis for $\text{col } U = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array} \right) \Rightarrow \begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 + x_4 &= 0 \end{aligned}$$

$$\Rightarrow \begin{aligned} x_1 &= -x_3 - x_5 \\ x_2 &= -x_4 \end{aligned}$$

$$\text{So Nul } U = \left\{ \begin{pmatrix} -x_3 - x_5 \\ -x_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \right\} = \left\{ x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

∴ Basis for $\text{Nul } U = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(21) Bases for col U :

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} \pi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 37 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 6 \end{pmatrix} \right\}$$

(or any set of 2 indpt vectors in \mathbb{R}^2).

$$\text{Row } U = \text{col}(U^T)$$

$$U^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{col}(U^T) \text{ has bases: } \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(26) S subspace of \mathbb{R}^6 , $\dim S = 5$.

(A) True. Let $B =$ basis for S .

Then B is a lin indpt. set

so B can be extended to a basis for \mathbb{R}^6 (by ZL).

(B) False (give counterexample)

(key is that S is fixed beforehand)

(30) 2nd & 3rd cols. have pivots

$$\text{so } \left\{ \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ is basis for col } U$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$