Math 102
Winter '08
Homework #2

$\boxed{1.5}$

① An upper-$\Delta$ matrix is nonsingular when it has no zeros on the diagonal.

④ • $A = \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix} \xrightarrow[m_{21}=4]{} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \Rightarrow A = \underbrace{\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}}_{U}$

• $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow[\substack{m_{21}=1/3 \\ m_{31}=1/3}]{} \begin{pmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 2/3 & 8/3 \end{pmatrix} \xrightarrow[m_{32}=1/4]{} \begin{pmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 0 & 15/6 \end{pmatrix}$

$\therefore A = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/3 & 1/4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 0 & 5/2 \end{pmatrix} = LU.$

• $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{pmatrix} \xrightarrow[\substack{m_{21}=1 \\ m_{31}=1}]{} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 4 & 7 \end{pmatrix} \xrightarrow[m_{32}=4/3]{} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{pmatrix}$

$\therefore A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4/3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{pmatrix} = LU.$

(15) • $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix}$ $\quad PA = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}$

$$PA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix} \xrightarrow[\substack{m_{21}=0 \\ m_{31}=2}]{} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{pmatrix} \xrightarrow[m_{32}=3]{} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\therefore PA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = LU$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = LDU$$

• $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ $\quad PA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{pmatrix}$

$$PA = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{pmatrix} \xrightarrow[\substack{m_{21}=1 \\ m_{31}=2}]{} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore PA = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = LU$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = LDU.$$

$\boxed{1.6}$

(4) (A) A invertible $\therefore$ There exists $A^{-1}$ s.t. $A^{-1}A = I$.

Given $AB = AC$

$$\Rightarrow A^{-1}(AB) = A^{-1}(AC)$$

$$\Rightarrow (\underbrace{\cancel{A^{-1}A}}_{I})B = (\underbrace{\cancel{A^{-1}A}}_{I})C$$

$$\Rightarrow B = C$$

(B)

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -5 & \pi \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$$

(6) $\underline{\underline{A_1}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ : $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$

$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$ $\therefore (A_1)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$\underline{\text{check}}$ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$ ✓

$\underline{\underline{A_2}} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ : $\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$

$\to \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 2 & 0 \\ 0 & 0 & 4 & 1 & 2 & 3 \end{array}\right)$

$\to \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 3/2 & 3 & 3/2 \\ 0 & 0 & 4 & 1 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 6 & 0 & 3 & 6 & 3 \\ 0 & 0 & 4 & 1 & 2 & 3 \end{array}\right)$

1.6 cont'd 

$$\rightarrow \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & 1 & 1/2 \\ 0 & 6 & 0 & 3 & 6 & 3 \\ 0 & 0 & 4 & 1 & 2 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right)$$

$$\therefore (A_2)^{-1} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix}$$

<u>check</u>: $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

$\underline{A_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ : $\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\therefore (A_3)^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

<u>check</u>: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

(22) $A = \begin{pmatrix} 0 & 3 \\ 4 & 6 \end{pmatrix} \Rightarrow A^{-1} = \dfrac{1}{0(6)-3(4)}\begin{pmatrix} 6 & -3 \\ -4 & 0 \end{pmatrix} = -\dfrac{1}{12}\begin{pmatrix} 6 & -3 \\ -4 & 0 \end{pmatrix}$

<u>check</u> $\begin{pmatrix} 0 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} -1/2 & 1/4 \\ 1/3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$B = \begin{pmatrix} a & b \\ b & 0 \end{pmatrix} \Rightarrow B^{-1} = \dfrac{1}{a(0)-b(b)}\begin{pmatrix} 0 & -b \\ -b & a \end{pmatrix} = -\dfrac{1}{b^2}\begin{pmatrix} 0 & -b \\ -b & a \end{pmatrix}$

$C = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} \Rightarrow C^{-1} = \dfrac{1}{7(3)-4(5)}\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}.$

 

(35) • $(A\ I) = \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right)$

$\therefore A^{-1} = \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix}$ check. $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

• $(A\ I) = \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -3 & -3 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & -1/3 \end{array}\right)$

$\to \left(\begin{array}{cc|cc} 1 & 0 & -3 & 4/3 \\ 0 & 1 & 1 & -1/3 \end{array}\right)$ $\therefore A^{-1} = \begin{pmatrix} -3 & 4/3 \\ 1 & -1/3 \end{pmatrix}$

check: $\begin{pmatrix} 1 & 4 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} -3 & 4/3 \\ 1 & -1/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

(50) $AB = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$

$(AB)^T = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$

$B^T A^T = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} = (AB)^T$

$A^T B^T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix} \neq (AB)^T$

$\boxed{11.7}$

③ $-\frac{d^2u}{dx^2} = f(x)$ $\quad \left.\frac{du}{dx}\right|_{x=0} = 0 \Rightarrow u_0 = u_1$, $\left.\frac{du}{dx}\right|_{x=1} \Rightarrow u_6 = u_5$.

Get system of equations:

$$-u_{j+1} + 2u_j - u_{j-1} = h^2 f(jh) \quad j = 1, \ldots, 5$$

$\underline{j=1}$: $-u_2 + \cancel{u_1} - \overset{0}{\cancel{u_0}} = h^2 f(h)$

$\underline{j=2}$: $-u_3 + 2u_2 - u_1 = h^2 f(2h)$

$\underline{j=3}$: $-u_4 + 2u_3 - u_2 = h^2 f(3h)$

$\underline{j=4}$: $-u_5 + 2u_4 - u_3 = h^2 f(4h)$

$\underline{j=5}$: $-\underset{u_5}{\cancel{u_6 +}} 2u_5 - u_4 = h^2 f(5h)$

So $A_0 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$, $A_0 \begin{pmatrix} c \\ c \\ c \\ c \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \forall\ c$

$\therefore A_0$ is singular

⑥ $u(0) = 1$, $u(1) = 0$

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = h^2 \begin{pmatrix} f(h)+1 \\ f(2h) \\ f(3h) \\ f(4h) \\ f(5h) \end{pmatrix}$$

1.7
#9

```
To get started, select MATLAB Help or Demos from the Help menu.

>> H=H(10)
??? Undefined function or variable "H".

>> H=matrixH(10)

H =

    1.0000    0.5000    0.3333    0.2500    0.2000    0.1667    0.1429    0.1250    0.1111    0.1000
    0.5000    0.3333    0.2500    0.2000    0.1667    0.1429    0.1250    0.1111    0.1000    0.0909
    0.3333    0.2500    0.2000    0.1667    0.1429    0.1250    0.1111    0.1000    0.0909    0.0833
    0.2500    0.2000    0.1667    0.1429    0.1250    0.1111    0.1000    0.0909    0.0833    0.0769
    0.2000    0.1667    0.1429    0.1250    0.1111    0.1000    0.0909    0.0833    0.0769    0.0714
    0.1667    0.1429    0.1250    0.1111    0.1000    0.0909    0.0833    0.0769    0.0714    0.0667
    0.1429    0.1250    0.1111    0.1000    0.0909    0.0833    0.0769    0.0714    0.0667    0.0625
    0.1250    0.1111    0.1000    0.0909    0.0833    0.0769    0.0714    0.0667    0.0625    0.0588
    0.1111    0.1000    0.0909    0.0833    0.0769    0.0714    0.0667    0.0625    0.0588    0.0556
    0.1000    0.0909    0.0833    0.0769    0.0714    0.0667    0.0625    0.0588    0.0556    0.0526
```

I had Matlab generate the Hilbert matrix (ASK me if you're not sure how to do this)

```
>> b=[1;0;0;0;0;0;0;0;0;0]

b =

     1
     0
     0
     0
     0
     0
     0
     0
     0
     0
```

Here I entered in the b vector and the error vector, e.

```
>> e=[.1;0;0;0;0;0;0;0;0;0]
```

t

```
e =

    0.1000
         0
         0
         0
         0
         0
         0
         0
         0
         0

>> H\b

ans =

  1.0e+006 *

    0.0001
   -0.0050
    0.0792
   -0.6006
    2.5226
   -6.3065
    9.6099
   -8.7519
    4.3760
   -0.9238
```

Here's the computed solution to $Hx = b$.

```
>> H\(b+e)

ans =

  1.0e+007 *

    0.0000
   -0.0005
```

Here's the computed solution to $Hy = b+e$

```
   0.0087
  -0.0661
   0.2775
  -0.6937
   1.0571
  -0.9627
   0.4814
  -0.1016

>> cond(H)

ans =

1.6025e+013

>>
```

HUGE number!

Why are the 2 computed solutions so different?

Because H is very ill-conditioned!

( cond(H) tells Matlab to calculate the condition # of H)

2.1 

③ $A=\begin{pmatrix}1&-1\\0&0\end{pmatrix}$ $\quad \text{col } A=\text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix}\right\}$

$$=\text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$$

$$=\boxed{\left\{\begin{pmatrix}c\\0\end{pmatrix} : c\in\mathbb{R}\right\}}$$

<u>nul A</u>: $\left(\begin{array}{cc|c}1&-1&0\\0&0&0\end{array}\right) \Rightarrow x_1-x_2=0 \Rightarrow x_1=x_2$

$\therefore$ nul $A=\left\{\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}x_1\\x_1\end{pmatrix}\right\}$

$$=\boxed{\left\{c\begin{pmatrix}1\\1\end{pmatrix} : c\in\mathbb{R}\right\}}.$$

$B=\begin{pmatrix}0&0&3\\1&2&3\end{pmatrix}$ $\quad \text{col } B=\text{span}\left\{\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}3\\3\end{pmatrix}\right\}$

$$=\text{span}\left\{\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}3\\3\end{pmatrix}\right\}$$

$$=\boxed{\left\{a\begin{pmatrix}0\\1\end{pmatrix}+b\begin{pmatrix}3\\3\end{pmatrix} : a,b\in\mathbb{R}\right\}}$$

<u>nul B</u>: $\left(\begin{array}{ccc|c}0&0&3&0\\1&2&3&0\end{array}\right)\to\left(\begin{array}{ccc|c}0&0&0&0\\1&2&3&0\end{array}\right)$

$\Rightarrow x_1+2x_2+3x_3=0 \Rightarrow x_1=-2x_2-3x_3$

$\therefore$ nul $B=\left\{\begin{pmatrix}-2x_2-3x_3\\x_2\\x_3\end{pmatrix}\right\}=\boxed{\left\{a\begin{pmatrix}-2\\1\\0\end{pmatrix}+b\begin{pmatrix}-3\\0\\1\end{pmatrix} : a,b\in\mathbb{R}\right\}}$

$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\boxed{\text{col } C = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}}$

$2 \times 3$

$\boxed{\text{nul } C = \mathbb{R}^3}$

⑦

Ⓐ Not a subspace b/c not closed under vector addition:

$$\underbrace{(1,0,1,0,\ldots)}_{\in S} + \underbrace{(0,1,0,1,\ldots)}_{\in S} = \underbrace{(1,1,1,\ldots)}_{\notin S}$$

Ⓑ Is a subspace b/c:

- Let $x = (x_1, x_2, \ldots)$ with $x_j = 0$ for all $j \geq J$
  $y = (y_1, y_2, \ldots)$ with $y_j = 0$ for all $j \geq K$.

  Then $x + y = (x_1 + y_1, x_2 + y_2, \ldots)$

  Let $M = \max\{J, K\}$

  Then $(x+y)_j = 0$ for all $j \geq M$

  $\therefore$ $x + y \in S$. $\Rightarrow$ $S$ closed under addition.

- Let $x = (x_1, x_2, \ldots) \in S$ with $x_j = 0$ $\forall j \geq J$

  "for all"

  Let $c \in \mathbb{R}$

  Then $cx = (cx_1, cx_2, \ldots)$ and $cx_j = 0$ $\forall j \geq J$

  $\therefore$ $cx \in S$ $\Rightarrow$ $S$ closed under scalar mult.

- $S$ obviously contains the zero vector $(0,0,0,\ldots)$

(c) Not a subspace:

$x = (-1, -2, -3, \ldots) \in S$, but $-1 \cdot x = (1, 2, 3, \ldots) \notin S$

$\therefore$ $S$ not closed under scalar multiplication.

(D) Is a subspace:

- $(x_1, x_2, \ldots) \in S$ with $\lim_{j \to \infty} x_j = x$

  $(y_1, y_2, \ldots) \in S$ with $\lim_{j \to \infty} y_j = y$

  Then $(x_1 + y_1, x_2 + y_2, \ldots)$ has

  $$\lim_{j \to \infty} (x_j + y_j) = x + y$$

  $\therefore$ $(x_1 + y_1, x_2 + y_2, \ldots) \in S$

  $\Rightarrow S$ closed under vector addition

- $(x_1, x_2, \ldots) \in S$ with $\lim_{j \to \infty} x_j = x$

  $\Rightarrow c(x_1, x_2, \ldots) = (cx_1, cx_2, \ldots)$ has

  $$\lim_{j \to \infty} cx_j = c \cdot \lim_{j \to \infty} x_j = cx$$

  $\therefore$ $c(x_1, x_2, \ldots) \in S$

  $\Rightarrow S$ closed under scalar multiplication.

- $(0, 0, \ldots) \in S$ since $\lim_{j \to \infty} 0 = 0$

(E) Is a subspace:

- Contains the $\mathbf{0}$ vector $(0, 0, \ldots)$
- $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots) \in S$

$\Rightarrow x_{j+1} - x_j$ same $\forall j$
$y_{j+1} - y_j$ same $\forall j$

and $x + y = (x_1 + y_1, x_2 + y_2, \ldots)$ has
$(x_{j+1} + y_{j+1}) - (x_j + y_j) = (x_{j+1} - x_j) + (y_{j+1} - y_j)$
same $\forall j$

$\therefore x + y \in S \Rightarrow S$ closed under addition

- You check scalar multiplication.

(F) Not a subspace. (check that $S$ is not closed under vector addition).

(8) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$

$\Rightarrow x_1 + x_2 + x_3 = 0, \quad x_1 + 2x_3 = 0$

$\Rightarrow x_1 = -2x_3, \quad -2x_3 + x_2 + x_3 = 0$

$\Rightarrow x_1 = -2x_3, \quad x_2 - x_3 = 0 \Rightarrow x_2 = x_3$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ x_3 \\ x_3 \end{pmatrix} = \left\{ c \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} : c \in \mathbb{R} \right\}$

This solution set represents a $\boxed{\text{line}}$ in $\mathbb{R}^3$, in the direction of $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

The solution set also represents a $\boxed{\text{subspace}}$ since:

- $\alpha \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = (\alpha + \beta) \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \in S.$ ✓
- $v \in S \Rightarrow v = \alpha \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

  Then $cv = c\left[\alpha \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right] = (c\alpha)\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \in S$ ✓
- $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \in S$ ✓

(25) If we add an extra column $b$ to a matrix $A$, then the column space gets larger unless $\underline{b \in col(A)}$. So,

$Ax = b$ has a solution

$\Leftrightarrow b \in col(A)$

$\Leftrightarrow col(A)$ <u>doesn't</u> get larger by including $b$.

<u>ex</u>: $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

If we augment $b = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $col(A)$ stays same.

If we augment $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $col(A)$ gets larger.

(26) Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Then $AB = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$

$\therefore$ $\text{col}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} = \mathbb{R}^2$

$\text{col}(AB) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \neq \mathbb{R}^2$