Last lecture: derived heat eqn for rod w/o heat source

heat equation with heat source: $Q(x,t) =$ heat energy generated by per unit volume per unit time

~~same derivation as before~~ except.

proceed as before:

$$\frac{\partial H}{\partial t} = \left[\phi(a,t) - \phi(b,t)\right]A + \int_a^b Q(x,t) A\,dx$$

heat exchange at boundaries

heat source additional term

$$= \frac{\partial}{\partial t}\int e(x,t) A\,dx$$

$$\Rightarrow \int_a^b \frac{\partial e}{\partial t}(x,t) A\,dx = \cancel{-\int_a^b \frac{\partial \phi}{\partial x}(x,t) A\,dx + \int_a^b Q(x,t)}$$

$$= \int_a^b \left[-\frac{\partial \phi}{\partial x}(x,t) + Q(x,t)\right] A\,dx$$

true for all $a<b$

$$\Rightarrow \frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$$

$e(x,t) = c(x)s(x)u(x,t)$

Fourier's law $\Rightarrow$ $\phi = -K_0(x)\frac{\partial u}{\partial x}$

general case of 1-dim heat equation

$$\boxed{c(x)s(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + Q}$$

$$\left(K_0(x)\frac{\partial u}{\partial x}\right)$$

Special Case of

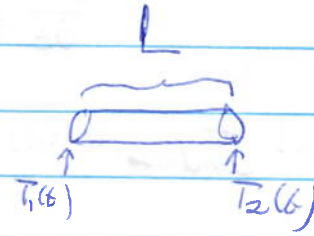
~~We derived~~ ~~the~~ heat equation: $\frac{\partial u}{\partial t} = \underset{\uparrow \text{k constant}}{k} \frac{\partial^2 u}{\partial x^2}$ for a one-dim rod.

temperature $u(x,t)$ depends on initial cond.

$$u(x,0) = f(x)$$

and on boundary conditions at each end

$$u(0,t) = T_1(t)$$
$$u(L,t) = T_2(t)$$



Special Case Equilibrium c temp distribution

- assume ~~temperatures constant at boundaries~~ $u(0,t)=T_1$, $u(L,t)=T_2$ $\forall t \geq 0$

* Equilibrium solution ~~mean~~ = solution with constant temperature $\Rightarrow \frac{\partial u}{\partial t} = 0$

$\Rightarrow$ solution of dif. equ. $\frac{d^2 u}{dx^2} = 0$ (u only depends on x)

$\Rightarrow u(x) = \cancel{ax+b}\ C_1 x + C_2$

Example ① Assume temperatures constant at boundaries, $\boxed{u(0,t)=T_1,\ u(L,T)=T_2 \text{ for all } t \geq 0}$

$$\left.\begin{aligned} T_1 &= u(0) = C_1 \cdot 0 + C_2 = C_2 \\ T_2 &= u(L) = C_1 L + C_2 \end{aligned}\right\} \quad \boxed{C_2 = T_1}$$

$T_2 = C_1 L + T_1$

$\Rightarrow \boxed{C_1 = \frac{T_2 - T_1}{L}}$

Solution: ~~$u(x)$~~ $\boxed{u(x,t) = \frac{T_2 - T_1}{L} x + T_1}$

Ex 2 Insulated boundaries

(*) $\frac{\partial u}{\partial x}(0,t)=0 \qquad \frac{\partial u}{\partial x}(L,t)=0 \quad \forall t$

as before, assume equilibrium state $\frac{\partial u}{\partial t}=0$, we have

$$u(x) = C_1 x + C_2$$

$$\left.\begin{aligned} \Rightarrow 0 &= u'(0) = C_1 \\ &(*) \\ 0 &= u'(L) = C_1 \end{aligned}\right\} \boxed{u(x) = C_2}$$

need additional information for calculating C_2
e.g. from initial condition $u(x,0) = f(x) = C_2$

$ Equilibrium ~~states~~ distributions do occur as limiting cases under stable conditions

Example: consider rod w/o heat sources
perfectly insulated

physics suggests: whatever initial conditions may have been, temperature distribution will eventually be constant over whole rod. ℓ

In mathematical terms:

~~General Case~~: If both ends of rod insulated

$\Rightarrow$ temperatures even out

$$\Rightarrow \quad \lim_{t\to\infty} u(x,t) = \underbrace{C_2}_{\text{for all } x} \quad \text{constant}$$

how to determine C_2?

Sol. use initial cond. $u(x,0) = f(x)$

heat energy $H(0) = \int_0^L c\rho\, u(x,0)\,dx = \int_0^L c\rho\, f(x)\,dx$

rod perfectly insulated $\Rightarrow$ $H(t) = \text{const.} = H(0) \quad \forall t$

$$\Rightarrow \quad \cancel{C_2 = \lim_{t\to\infty}} \; u$$

$$\int_0^L c\rho\, f(x)\,dx = H(0) = \lim_{t\to\infty} H(t) = \lim_{t\to\infty} \int c\rho\, \underbrace{u(x,t)}_{\to C_2}\,dx$$

$$= \int_0^L c\rho\, C_2\,dx = c\rho L C_2$$

$$\Rightarrow \boxed{C_2 = \frac{1}{L}\int_0^L f(x)\,dx}$$

Result: For a perfectly insulated rod, the temperature $u(x,t)$ will go to the average temperature $\frac{1}{L}\int f(x)\,dx$ at time $t=0$