3)

Remark Let V, W be G-modules

$\Rightarrow$ $\mathrm{Hom}(V,W)$ is a G-module with G action defined by

$$g\cdot T = \rho_W(g)\, T\, \rho_V(g)^{-1} \qquad \text{where } \rho_W(g): w\in W \to g.w$$

If G Lie group, $X \in \mathfrak{g}$

$$\Rightarrow \quad X\cdot T = \frac{d}{dt}\left(\exp(tX)\, T \exp(-tX)\right)\Big|_{t=0}$$

$$= XT - TX$$

Let $\mathrm{Hom}_{\mathfrak{g}}(V,W) = \{T: V\to W,\ XT(v) = TX(v)\ \forall X\in\mathfrak{g}, v\in V\}$

$\Rightarrow$ action of $\mathfrak{g}$ on $\mathrm{Hom}_{\mathfrak{g}}(V,W)$ trivial

Theorem $\mathfrak{g}$ s.s. $W\subset V$ $\mathfrak{g}$-module $\Rightarrow$ $\exists$ $\mathfrak{g}$-module $W'\subset V$ s.t. $V = W\oplus W'$

Proof. (a) assume W irred, V/W 1-dim.

$\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ $\Rightarrow$ $\mathfrak{g}$ acts trivially on V/W

$\Rightarrow$ C_V acts as 0 on V/W.

$\mathrm{Tr}(C_V) = \dim\mathfrak{g}$ $\Rightarrow$ C_V acts via nonzero scalar on W

Let W' = eigenspace of C_V for eigenvalue 0

$\Rightarrow$ $V = W\oplus W'$ as $\dim W'\geq 1$, $\dim W = \dim V - 1$ and $W\cap W' = 0$

(b) statement in (a) also holds for arbitrary submodule $W\subset V$, $\dim V/W = 1$

proof. Let $0\subset W_1\subset W_2\subset \dots \subset W_k = W$ $\mathfrak{g}$-modules s.t. W_{i+1}/W_i simple

proof by ind. on k. by ind. ass. $\exists\ \tilde{W}'$ s.t.

$$V/W_1 \cong W/W_1 \oplus \tilde{W}'/W_1 \qquad (*)$$

by (a) $\tilde{W}' = W_1 \oplus W'$ for a $\mathfrak{g}$-submodule W'

$$\Rightarrow W'\cap W = (W'\cap\tilde{W}')\cap W = W'\cap(\tilde{W}'\cap W) \underset{(*)}{=} W'\cap W_1 = 0$$

(c) $W\subset V$ irred. V/W arbitrary.

Consider the map $\rho: \mathrm{Hom}_{\mathfrak{g}}(V,W) \to \mathrm{Hom}_{\mathfrak{g}}(W,W)$, $T \mapsto T|_W$

this is a $\mathfrak{g}$-module map (action is trivial on both $\mathrm{Hom}_{\mathfrak{g}}(V,W)$ and $\mathrm{Hom}_{\mathfrak{g}}(W,W)$)

image of $\rho = \mathrm{Hom}_{\mathfrak{g}}(W,W) = \mathbb{C}1_W$

$\ker\rho$ is submodule of $\mathrm{Hom}_{\mathfrak{g}}(V,W)$ with codimension 1

(a)&(b) $\Rightarrow$ $\mathrm{Hom}_{\mathfrak{g}}(V,W) = \ker\rho \oplus \mathbb{C}\psi$, $\psi|_W = \mathrm{id}_W$

$\Rightarrow$ $V = \ker\psi \oplus W$ (if $v\in\ker\psi\cap W$ $\Rightarrow$ $v \underset{v\in W}{=} \psi(v) \underset{v\in\ker\psi}{=} 0$)