SOLUTIONS OF SOLUTIONS OF OLD EXAM PROBLEMS

READ THIS: Unfortunately, the solutions on the webpage from which I took these old exam problems referred to another exam. Below you can find SKETCHES of solutions. These are less detailed than you should do it in the exam due to lack of time. I still hope this will be useful. If you need more information, come to my office hour Tuesday, 3-4.

0.1. **Problem 1.** We have to solve the differential equation $dy/dx = 2x$ which has the solution $y = x^2$. For part (a) we obtain $u(x, y) = (y - x^2)^2$. For part (b), we would get

$$\sin(x) = u(x, 0) = g(y - x^2).$$

For $x > 0$, this would imply $u(x, y) = \sin(\sqrt{x^2 - y})$, while for $x < 0$ we would get $u(x, y) = \sin(-\sqrt{x^2 - y})$. Obviously, these are two different functions. Hence there is no solution whose domain of definition would contain the whole x-axis.

0.2. **Problem 2.** Observe that

$$u_x(x, 0) = \cos(x) + 6\cos(3x) - 30\cos(6x) + 11\cos(11x).$$

Using orthogonality $\int_0^\pi \sin(nx)\sin(mx)\ dx = 0$ if $n \neq m$ and $\int_0^\pi \sin^2(nx)\ dx = \pi/2$ (and same for cosines), it is easy to calculate

$$E(0) = \frac{\pi}{4}[1 + 6^2 + 30^2 + 11^2 + 1 + 16 + 1 + 9] = 1085.$$

As the energy for a solution of the wave equation is independent of time, this is the solution in general.

0.3. **Problem 4.** This was done in class, except for (c). I was a little bit sloppy for part (a), so let me do this again here: If $\Delta u = 0$, we get

$$0 = \int\int u_{xx} + u_{yy}\ dxdy.$$

We know calculate

$$\int_0^b \int_0^a u_{xx}\ dxdy = \int_0^b [u_x(x, y)]_{x=0}^{x=a} dy = \int_0^b [u_x(a, y) - u_x(0, y)]dy.$$

Similarly, after now integrating the y-coordinate first, we get

$$\int_0^a \int_0^b u_{yy}\ dydx = \int_0^a [u_y(x, b) - u_y(x, 0)]dx.$$

Adding both the left hand sides and the right hand sides of the two equations above, and plugging in the functions of the boundary conditions on the right hand sides proves the desired equality in (a).

0.4. **Problem 5.** A similar problem was also done in class (look at your notes!). Following the hint, we consider two Dirichlet boundary problems. In the first one, the boundary conditions are the same as the given one, except that $u_1(x,1) = 1$. In the second problem we have $u_2(1,x) = \sin(\pi x)$ and all the other boundary conditions are equal to 0. Then it is easy to check that $u_1(x,y) = y$ is the solution of the first problem.

For the second problem, we try $u_2(x,y) = Y(y)\sin(\pi x)$. Hence we get $Y''/Y = \pi^2$, i.e. $Y(y) = c_1\cosh(\pi y)+c_2\sinh(\pi y)$. The boundary conditions $u_2(x,0) = 0$ and $u_2(x,1) = \sin(\pi x)$ translate to $Y(0) = 0$ and $Y(1) = 1$. Hence $c_1 = 0$ and $c_2 = 1/(\sinh(\pi))$. Hence

$$u_2(x,y) = \frac{\sin(\pi x)\sinh(\pi y)}{\sinh(\pi)}.$$

So the complete solution is

$$u(x,y) = y + \frac{\sin(\pi x)\sinh(\pi y)}{\sinh(\pi)}.$$

0.5. **Problem 6.** We do the problem in polar coordinates. For a separable solution $R(r)\Theta(\theta)$, we get the equations

$$\Theta'' + \lambda\Theta = 0, \quad r^2R'' + rR' - \lambda R = 0.$$

As Θ has to satisfy the periodic boundary conditions $\Theta(0) = \Theta(2\pi)$, we obtain that $\lambda = n^2$, $n = 1, 2, \ \ldots$; in particular, it is positive. Plugging this into the differential equation for R, we obtain $R(r) = c_1r^n + c_2r^{-n}$. As our solutions should be well-defined for $r = 0$, we only use the positive exponents. Hence the general solution is

$$u(r,\theta) = A_0/2 \ + \ \sum_{n=1}^{\infty} r^n(A_n\cos(n\theta) + B_n\sin(n\theta)).$$

The solution for our given boundary value is

$$u(r,\theta) = 1 + r^2(\cos(2\theta) + \sin(2\theta)).$$

To check the maximum principle, observe that $u(r,\theta) - 1 = r^2(\cos(2\theta) + \sin(2\theta))$. Hence for any given fixed θ_0 the function $|u(r,\theta)| = r^2|\cos(2\theta_0) + \sin(2\theta_0)|$ will have its maximum at $r = 1$, i.e. on the boundary of the disk of radius 1 around the origin.

0.6. **Problem 2.** We use d'Alembert's formula for the wave equation $u_{tt} = u_{xx}$ with initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$. This states that the solution is equal to

$$u(x,t) = \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2}\int_{x-t}^{x+t}\psi(s)ds.$$

If $x = t = 10$, we get $\phi(x+t) = \phi(20) = 0$. Hence

$$u(10,10) = \frac{1}{2}(\phi(0) + \int_0^{20}\psi(s)\ ds) =$$

$$\frac{1}{2}(\cos(0) + \int_0^{\pi}\sin(s)\ ds) = \frac{1}{2}(1+2) = \frac{3}{2}.$$