MATH 132
EXAM #2

Please answer the following questions. Because this test is open book and open note, you *will not get credit* for answers unless you demonstrate how you arrived at them. In short, please show all work.

Problem 1 (10 pts)

Consider the three dimensional Poisson Kernel:

$$K(\vec{x}_0, \vec{x}) \ = \ \frac{a^2 - |\vec{x}_0|^2}{4\pi a} \cdot \frac{1}{|\vec{x} - \vec{x}_0|^3} \ .$$

Please answer the following:

- For a fixed vector $\vec{x} \in \partial B_a$ (i.e. such that $|\vec{x}| = a$), compute the integral:

  $$I(b) \ = \ \iint_{|\vec{x}_0| = b} K(\vec{x}_0, \vec{x}) \ dS_{\vec{x}_0} \ ,$$

  where $b < a$ is some fixed constant. (Hint: This is probably a *quite* difficult problem to do in terms of an explicit calculation. Instead, use the fact that K is harmonic in the $\vec{x}_0$ variable, and simply rely on things you know about harmonic functions to get the answer.)

- Show that $\lim_{b \to a} I(b) = 1$. That is, the total integral of K with respect to the $\vec{x}_0$ variable on spheres approaching radius a converges to 1. (This shows that the Poisson Kernel approaches a "delta mass" supported at $\vec{x}$ on ∂B_a as one moves out in the $\vec{x}_0$ variable.)

Problem 2 (10 pts)

Consider the solution u to the two dimensional wave equation:

$$\begin{aligned} \partial_t^2 u - \Delta u &= 0 \ , \\ u(0,\vec{x}) &= 0 \ , \\ u_t(0,\vec{x}) &= e^{-|\vec{x}|^2} \ . \end{aligned}$$

Please answer the following:

- Show that the local energy in the unit disk centered at the origin at $t = 1$ is less than $\frac{\pi}{4}(1 - e^{-8})$. That is, prove the bound:

$$\frac{1}{2}\iint_{x^2+y^2\leqslant 1} \big(u_t^2 + |\nabla u|^2\big)(1,x,y)\ dxdy \ \leqslant\ \frac{\pi}{4}(1-e^{-8}) \ .$$

  (Hint: It is not hard to integrate a 2D Gaussian, i.e. $e^{-|\vec{x}|^2}$, using polar coordinates.)

- Conclude from this bound and the conservation of total energy that one also has the following *lower* bound on the energy *outside* the unit disk at time $t = 1$:

$$\frac{\pi}{4}e^{-8} \ \leqslant\ \frac{1}{2}\iint_{x^2+y^2>1} \big(u_t^2 + |\nabla u|^2\big)(1,x,y)\ dxdy \ .$$

Problem 3 (10 pts)

Consider the solution u to the <u>three dimensional</u> wave equation:

$$\begin{aligned} \partial_t^2 u - c^2 \Delta u &= 0 , \\ u(0, \vec{x}) &= 0 , \\ u_t(0, \vec{x}) &= \begin{cases} 1 - |\vec{x}|^2 , & \text{if } |\vec{x}| \leqslant 1 ; \\ 0 , & \text{otherwise} . \end{cases} \end{aligned}$$

Give a precise lower bound $0 < T$ such that for all $T < t$ one has the condition:

$$u(t, 0, 0, 0) = 0 .$$

That is, compute the *smallest* positive time T such that the wave u vanishes at the origin after that time. Please give the precise reason for your answer. (Hint: Don't forget the c^2 in front of this wave equation!)

The following problem will show that $2D$ waves decay only like $O(t^{-\frac{1}{2}})$ "along the cone".

Extra Credit Problem (10 pts)

Consider the solution u to the <u>two dimensional</u> wave equation:

$$\begin{aligned} \partial_t^2 u - c^2 \Delta u &= 0 \, , \\ u(0, \vec{x}) &= 0 \, , \\ u_t(0, x, y) &= \begin{cases} 1 \, , & \text{in the box } \ |x|, |y| \leqslant 1 \, ; \\ 0 \, , & \text{otherwise} \, . \end{cases} \end{aligned}$$

Here is a picture of where the initial data $u_t(0, x, y)$ is 1 (the shaded region):

Show that if one goes out along the ray from the origin along the x-axis with velocity c, then the infinite limit of $t^{\frac{1}{2}} u$ along this ray is equal to $\sqrt{2} c^{-\frac{3}{2}} \pi^{-1}$. That is, use the Kirchhoff formula to compute:

$$\lim_{t \to \infty} t^{\frac{1}{2}} u(t, ct, 0) = \frac{2^{\frac{1}{2}}}{c^{\frac{3}{2}} \pi} \, .$$

In particular, u can decay no better than a constant times $t^{-\frac{1}{2}}$ in this direction.

(Hints: First of all, it helps to draw the limiting region in the x,y-plane where you need to integrate in Kirchhoff's formula as $t \to \infty$. In the limit, this should be some fixed portion of the box $|x|, |y| \leqslant 1$.

Next, you may assume that you can take limits under any integral sign in this problem. That is, just multiply the explicit Kirchhoff formula for this situation by $t^{\frac{1}{2}}$, and see if you can clean up the integrand in the limit as $t \to \infty$. You should be left with an (improper) integral that you can evaluate explicitly.)