OFFICIAL 'CHEAT SHEET' FOR MIDTERM II

Continuous, integrable and differentiable functions Recall that $f : [a, b] \to \mathbf{R}$ is called continuous at x_0 if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ if $|x - y| < \delta$.

We have the implications f differentiable $\Rightarrow$ f continuous $\Rightarrow$ f integrable.

Fundamental Theorems of Calculus (a) Assume that F is differentiable on (a, b) and continuous on $[a, b]$ such that also $F'(x)$ is continuous on $[a, b]$. Then

$$\int_a^b F'(x) \ = \ F(b) - F(a).$$

(b) Assume $f : [a, b] \to \mathbf{R}$ is continuous. Then

$$\frac{d}{dx}\left[\int_a^x f\right] = f(x), \qquad \frac{d}{dx}\left[\int_x^b f\right] = -f(x).$$

You are allowed to use integration by parts, if convenient.

Identity Criterion If I is an interval and $f, g : I \to \mathbf{R}$ are two differentiable functions satisfying $f'(x) = g'(x)$ for all $x \in I$, then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in I$.

Taylor polynomials and approximations Let I be an open interval and let $f : I \to \mathbf{R}$ be a function with n derivatives. Then its n-th Taylor polynomial p_n at $x_0 \in I$ is defined to be

$$p_n(x) \ = \ \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \ (x - x_0)^k.$$

Lagrange Remainder Theorem Assume $f : I \to \mathbf{R}$ has $n+1$ derivatives. Let $x_0, x \in I$. Then there exists a number c between x_0 and x such that

$$f(x) - p_n(x) \ = \ \frac{f^{(n+1)}(c)}{(n+1)!} \ (x - x_0)^{n+1}.$$

Lemma (a) Let c be a constant. Then we have

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0.$$

(b) Let (c_n) be a sequence such that $\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = r$.

(i) If $r < 1$, then $\lim_{n\to\infty} c_n = 0$

(ii) If $r > 1$, then (c_n) is an unbounded sequence.

Examples of Taylor series (a) If $|x| < 1$, we have

$$\ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}.$$

(b) Let $\beta \in \mathbf{R}$ and let $|x| < 1$. Then

$$(1 + x)^\beta = \sum_{k=0}^{\infty} \binom{\beta}{k} x^k.$$

Theorem Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then $f^{(k)}(0) = 0$ for all $k \in \mathbf{N}$, and f has derivatives of all orders for all $x \in \mathbf{R}$.