OFFICIAL 'CHEAT SHEET' FOR MIDTERM I

**Darboux sums and partitions** Let $f : [a, b] \to \mathbf{R}$ be a bounded function. For a partition $P = \{x_0 = a, x_1, \ ..., \ x_n = b\}$, where $x_i < x_i + 1$ for $0 \le i < n$, we have the *upper* and *lower Darboux sums*

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \quad \text{and} \quad L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1});$$

here $m_i$ and $M_i$ are infimum and supremum of $f$ on the interval $[x_{i-1}, x_i]$. A *refinement* $P^*$ of $P$ is a partition of the same interval which contains all the points of $P$.

**Lemma** (a) If there are numbers $m, M$ such that $m \le f(x) \le M$ for all $x \in [a, b]$, then $m(b - a) \le L(f, P)$ and $U(f, P) \le M(b - a)$ for any partition $P$ of $[a, b]$.

(b) (Refinement lemma) If $P^*$ is a refinement of $P$, then $L(f, P) \le L(f, P^*)$ and $U(f, P^*) \le U(f, P)$.

**Definition of integral** We define the upper and lower integrals by

$$\int_a^{\bar{b}} f = \inf U(f, P), \qquad \int_{\bar{a}}^{b} f = \sup L(f, P),$$

where the inf and sup are taken over all partitions of $[a, b]$. A bounded function is called *integrable* if upper and lower integrals coincide.

**Theorem** (a) The lower integral of $f$ is always less or equal than the upper integral.

(b) (Archimedes-Riemann) A bounded function $f : [a, b] \to \mathbf{R}$ is integrable if and only if there exists a sequence of partitions $(P_n)$ such that

$$\lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0.$$

**Properties of integrals** Let $f, g : [a, b] \to \mathbf{R}$ be integrable functions. Then we have

(a) $\int_a^b f = \int_a^c f + \int_c^b f$, for $a < c < b$,

(b) If $f(x) \le g(x)$ for all $x \in [a, b]$, then $\int_a^b f \le \int_a^b g$,

(c) If $\alpha$ and $\beta$ are numbers, then $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$.

**Continuous functions** Recall that $f : [a, b] \to \mathbf{R}$ is called continuous at $x_0$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ if $|x - y| < \delta$.

**Theorem** (a) All continuous functions are integrable.

(b) Let $f : [a, b] \to \mathbf{R}$ be a function which is continuous and bounded on $(a, b)$. Then $f$ is integrable, and the value of the integral $\int_a^b f$ does not depend on the values of $f$ at $a$ and $b$.