OFFICIAL 'CHEAT SHEET'

Darboux sums and partitions Let $f : [a, b] \to \mathbf{R}$ be a bounded function. For a partition $P = \{x_0 = a, x_1, \ ..., \ x_n = b\}$, where $x_i < x_i + 1$ for $0 \le i < n$, we have the *upper* and *lower Darboux sums*

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \quad \text{and} \quad L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1});$$

here m_i and M_i are infimum and supremum of f on the interval $[x_{i-1}, x_i]$.

Definition of integral We define the upper and lower integrals by

$$\int_a^{\bar{b}} f = \inf U(f, P), \qquad \int_{\bar{a}}^b f = \sup L(f, P),$$

where the inf and sup are taken over all partitions of $[a, b]$. A bounded function is called *integrable* if upper and lower integrals coincide.

Theorem (a) The lower integral of f is always less or equal than the upper integral.

(b) (Archimedes-Riemann) A bounded function $f : [a, b] \to \mathbf{R}$ is integrable if and only if there exists a sequence of partitions (P_n) such that

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Continuous, integrable and differentiable functions Recall that $f : [a, b] \to \mathbf{R}$ is called continuous at x_0 if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ if $|x - y| < \delta$.

Theorem (a) Every continuous function is integrable

(b) Assume $f : [a, b] \to \mathbf{R}$ is continuous and $f(x) \ge 0$ for all $x \in [a, b]$. If there exists $x_0 \in [a, b]$ such that $f(x_0) > 0$, then $\int_a^b f > 0$

We have the implications f differentiable $\Rightarrow$ f continuous $\Rightarrow$ f integrable.

Fundamental Theorems of Calculus (a) Assume that F is differentiable on (a, b) and continuous on $[a, b]$ such that also $F'(x)$ is continuous on $[a, b]$. Then

$$\int_a^b F'(x) \ = \ F(b) - F(a).$$

(b) Assume $f : [a, b] \to \mathbf{R}$ is continuous. Then

$$\frac{d}{dx}\left[\int_a^x f\right] = f(x), \qquad \frac{d}{dx}\left[\int_x^b f\right] = -f(x).$$

Taylor polynomials and approximations Let I be an open interval and let $f : I \to \mathbf{R}$ be a function with n derivatives. Then its n-th Taylor polynomial p_n at $x_0 \in I$ is defined to be

$$p_n(x) \ = \ \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \ (x - x_0)^k.$$

The function f is given by its Taylor series at x, i.e. $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} \ (x - x_0)^k$, if $f(x) = \lim_{n\to\infty} p_n(x)$.

Lagrange Remainder Theorem Assume $f : I \to \mathbf{R}$ has $n + 1$ derivatives. Let $x_0, x \in I$. Then there exists a number c between x_0 and x such that

$$f(x) - p_n(x) \ = \ \frac{f^{(n+1)}(c)}{(n + 1)!} \ (x - x_0)^{n+1}.$$

Lemma (a) Let c be a constant. Then we have

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0.$$

(b) Let (c_n) be a sequence such that $\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = r$.

(i) If $r < 1$, then $\lim_{n\to\infty} c_n = 0$

(ii) If $r > 1$, then (c_n) is an unbounded sequence.

Weierstrass Approximation Theorem Let $f : [a,b] \to \mathbf{R}$ be a continuous function, and let $\epsilon > 0$. Then there exists a polynomial p such that $|p(x) - f(x)| < \epsilon$ for all $x \in [a,b]$.

Pointwise and uniform convergence Let $f_n : D \to \mathbf{R}$ be a sequence of function, and let $f : D \to \mathbf{R}$.

(a) The sequence (f_n) converges to f *pointwise* if $\lim f_n(x) = f(x)$ for all $x \in [a,b]$.

(b) The sequence (f_n) converges to f *uniformly* if for every $\epsilon > 0$ we can find an N such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in [a,b]$ and all $n \geq N$.

Theorem Assume $f_n \to f$ uniformly, and $D = [a,b]$.

(a) If all f_ns are continuous, then so is f.

(b) If all f_ns are integrable, then so is f. Moreover, in this case $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.

(c) Assume all f_n's are differentiable. If the f_n's converge uniformly to a function g, and the functions f_n converge pointwise to the function f, then f is differentiable and $f' = g = \lim_{n\to\infty} f_n'$.

Theorem Assume for some $r > 0$ the function $f : (-r, r) \to \mathbf{R}$ is given by the power series

$$f(x) = \sum_{k=0}^{\infty} c_k x^k, \qquad \text{if } |x| < r.$$

Then f has derivatives of all orders. In particular

$$f'(c) = \sum_{k=0}^{\infty} k c_k x^{k-1}, \qquad \text{if } |x| < r.$$