SOLID VIBRATIONS IN A BALL

These are supplementary notes to chapters 10.3 and 10.6 in Strauss's book on PDE, second edition. *There is no new material in these notes.* I only somewhat rearranged the material, which makes it a little easier to follow (for me, at least). We use the conventions in the book. In particular, D is the ball of radius a, centered at the origin. We first consider the eigenvalue problem

$$-\Delta v = \lambda v, \tag{1}$$

without assuming any boundary conditions. We have the following crucial lemma:

Lemma 1 The eigenvalue problem has solutions of the form

$$v(r, \theta, \phi) = R(r)Y(\theta, \phi),$$

where

$$R(r) = \begin{cases} r^\ell & \text{if } \lambda = 0, \\ \frac{J_{\ell+\frac{1}{2}}(\sqrt{\lambda}r)}{\sqrt{r}} & \text{if } \lambda > 0, \end{cases}$$

and

$$Y(\theta, \phi) = Y_\ell^m(\theta, \phi) = P_\ell^{|m|}(\cos\theta)e^{im\phi}.$$

Here $\ell = 0, 1, 2, \ ..., -\ell \le m \le \ell$, and the P_ℓ^m are the associated Legendre functions (see Section 10.6 in the book or below).

Proof. This lemma is proved in the book on pages 271-274, together with some results on (associated) Legendre polynomials in Section 10.6. There is little point in retyping the details. We just note the following:

The proof is done, as usual, via separating variables, where we use spherical coordinates. The study of the $R(r)$ function can be reduced to our work on Bessel functions in the two-dimensional case, after some clever substitutions. One obtains that

$$R(r) = \frac{J_{\sqrt{\gamma+\frac{1}{2}}}(\sqrt{\lambda}r)}{\sqrt{r}}. \tag{2}$$

Similarly, it is fairly straightforward to show that $Y(\theta, \phi) = p(\theta)q(\phi)$ with

$$q(\phi) = A\cos m\phi + B\sin m\phi, \quad \text{for some } m = 0, 1, 2, \ ...$$

To determine $p(\theta)$, one makes the substitution $s = \cos\theta$. Using chain rule and $\sin^2\theta = 1 - s^2$, one separates the equation for p as the *associated Legendre equation*

$$[(1 - s^2)p']' = (\gamma - \frac{m^2}{1 - s^2})p = 0, \tag{AL}$$

where now differentiation is done with respect to the variable s, see p. 273. The proof will be finished after finding suitable solutions of the associated Legendre equation. Getting these solutions will force $\gamma = \ell(\ell+1)$, which entails $\gamma + \frac{1}{2} = (\ell + \frac{1}{2})^2$. Hence the index of the Bessel function in (2) will be as stated.

Remark We can also obtain power series solutions of (AL), for all values of γ. However, it is not hard to check, using the recursion relation (RC) below that these solutions only have radius of convergence 1. With a little more work, one can indeed show that these solutions would not be finite at $z = \pm 1$. They are therefore not useful for our purposes where we want our solutions to have finite values at the boundary.

We now formulate the Legendre equation denoting the function by $u(z)$. It is the special case of (AL) with $m = 0$. Hence we get

$$(1-z^2)u'' - 2zu' + \gamma u = 0. \qquad (L)$$

Plugging $u(z) = \sum_{k=0}^{\infty} a_k z^k$ into this equation, we get

$$\sum_k k(k-1)a_k z^{k-2} = \sum_k (k(k-1) + 2k - \gamma)a_k z^k.$$

Substituting $(k-2) \to k$ on the left hand side and comparing coefficients of z^k, we obtain that a_0, a_1 are arbitrary, and

$$(k+1)(k+2)a_{k+2} = (k(k+1) - \gamma)a_k. \qquad (RC)$$

Lemma 2 We only get polynomial solutions of (L) if $\gamma = \ell(\ell+1)$ for an integer $\ell \geq 0$. In this case, the unique polynomial solution, up to a scalar multiple, is given by the Legendre polynomial

$$P_\ell(z) = \frac{1}{2^\ell} \sum_{j=0}^{m} \frac{(-1)^j}{j!} \frac{(2\ell-2j)!}{(\ell-2j)!(\ell-j)!}\, z^{\ell-2j},$$

where $\ell = 2m$ or $\ell = 2m+1$ depending on the parity of ℓ.

Proof. It is clear from the recursion relation above that if γ is not as indicated, a_{k+2} will be a nonzero multiple of a_k for all k. Hence $u(z)$ can only have a finite number of nonzero coefficients if $\gamma = \ell(\ell+1)$. To prove that P_ℓ is a solution of (L), it suffices to check that the recursion relations (RC) are satisfied. Setting $k = \ell - 2j$, this means that

$$\frac{a_{\ell-2j+2}}{a_{\ell-2j}} = \frac{-j!(\ell-2j)!(\ell-j)!(2\ell-2j+2)!}{(j-1)!(2\ell-2j)!(\ell-2j+2)!(\ell-j+1)!} =$$

$$= -\frac{j(2\ell-2j+2)(2\ell-2j+1)}{(\ell-2j+2)(\ell_2 j+1)(\ell-j+1)} = -\frac{2j(2\ell-2j+1)}{(\ell-2j+2)(\ell_2 j+1)}.$$

Substituting back $2j = \ell - k$, one checks that the last expression is equal to

$$\frac{k(k+1) - \ell(\ell+1)}{(k+1)(k+2)}.$$

This proves that P_ℓ is a solution of Legendre's equation. It is not very hard to show that this is the only way to get a polynomial solution of Legendre's equation. We omit this here.

Remark: The formula above for Legendre polynomials is not very intuitive. For practical purposes, the following recursion relation, proved in the homework, is more convenient:

$$(\ell+1)P_{\ell+1}(z) - (2\ell+1)zP_\ell(z) + \ell P_{\ell-1}(z) = 0.$$

As $P_0(z) = 1$ and $P_1(z) = z$, one can inductively calculate the Legendre polynomials from the relation above. See p. 290 for a list of the first few Legendre polynomials. Check this by yourself.

We have found all polynomial solutions for Legendre's equation. It is now easy to extend this to get (almost) polynomial solutions of the *associated* Legendre equation (AL)

Lemma 3 If $u(z)$ is a solution of Legendre's equation (L), then $(1-z^2)^{m/2}u^{(m)}(z)$ is a solution of the associated Legendre equation (AL); here $u^{(m)}$ denotes the m-th derivative of u. In particular, we have the following solution, usually referred to as associated Legendre function

$$P_\ell^m(z) = (1-z^2)^{m/2} P_\ell^{(m)}(z).$$

Proof. The proof was done in class, and is also outlined in the book on p. 293. One first shows by induction on m that $v = u^{(m)}$ is a solution of the differential equation

$$(1-z^2)v'' - 2(m+1)zv' + [\gamma - m(m+1)]v = 0.$$

Substituting $v = (1-z^2)^{-m/2}w(z)$ into this equation and simplifying it shows that w satisfies (AL).

Remark: A list of the first few associated Legendre polynomials can be found on p. 276. It is instructive to check this directly from the material in these notes.

Also observe that while P_ℓ^m is not a polynomial for m odd (because of the factor $(1-z^2)^{m/2}$), the expression $P_\ell^m(\cos\theta)$, which occurs in Lemma 1, is a polynomial in $\cos\theta$ and $\sin\theta$.

SOLUTIONS OF INITIAL VALUE AND BOUNDARY VALUE PROBLEMS

We can now calculate these solutions in the usual way. Let us do one case here to show how the boundary conditions will determine possible values of λ, see the result (19) on p. 274. Let

$$0 < z_{\ell 1} < z_{\ell 2} < z_{\ell 3} < \ \dots$$

denote the zeros of the Bessel function $J_{\ell+1/2}$, and determine $\lambda_{\ell j}$ by

$$\sqrt{\lambda_{\ell j}}a = z_{\ell j},$$

where a is the radius of the ball D.

Lemma 4 Consider the boundary value problem $\Delta v = -\lambda v$ with boundary value $v = 0$ on the boundary of the ball D, i.e. for $r = a$, and v finite everywhere. Then these conditions force

$$\lambda = \lambda_{\ell j} \quad \text{for some } \ell, j,$$

and the solution is given by

$$v_{\ell,m,j}(r,\theta,\phi) \ = \ \frac{J_{\ell+\frac{1}{2}}(\sqrt{\lambda_{\ell j}}r)}{\sqrt{r}} \ P_\ell^{|m|}(\cos\theta)\ e^{im\phi},$$

where $-\ell \leq m \leq \ell$, $\ell = 0, 1, 2, \ \dots.$ and $j = 1, 2, \ \dots$.

Moreover, all these solutions are mutually orthogonal with respect to the inner product coming from integrating over the ball of radius a.

Proof. The first statement follows from Lemma 1 and the boundary condition, which implies $J_{\ell+\frac{1}{2}}(\sqrt{\lambda}a) = 0$. The second statement follows from the fact that eigenfunctions belonging to different eigenvalues have to be orthogonal, see Theorem 1 on p. 260. It is conceivable that $\lambda_{\ell' j'} = \lambda_{\ell j}$ for $\ell' \neq \ell$ and $j' \neq j$, so one needs to be a little more careful. But this can be dealt with with arguments as in the discussion on p. 263.

Having solutions of the boundary problem as above, we can now also find solutions of diffusion and wave equations over the ball of radius a in the usual way. So if v satisfies $\Delta v = -\lambda v$, we get a solution of the diffusion equation $u_t = k\Delta u$ of the form

$$u(\mathbf{x}, t) = v(\mathbf{x})e^{-k\lambda t},$$

and we get solutions of the wave equation $u_{tt} = c^2\Delta u$ of the form

$$u(\mathbf{x}, t) = v(\mathbf{x})(A\cos(\sqrt{\lambda}ct) + B\sin(\sqrt{\lambda}ct)).$$

This can be used in connection with Lemma 1 and Lemma 4 to get series solutions of boundary value problems, as in Example 1 on p. 262.

Remark The same strategy also works for other boundary conditions (see e.g. homework problem 10.3.4) which would usually force different allowable eigenvalues λ.