Some Solutions for Practice Final 2

Observe : Solutions here may sometimes be a little sketchier than what is expected in the exam. Make sure you justify all your steps.

1. (a) Calculate the coefficients using the formula (for $L = 1$)

$$A_n = 2 \int_0^1 x \cos n \pi x \, dx$$

via integration by parts.

(b) Using the fact that $x = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n \pi x$, we obtain

$$\int_0^1 x^2 \, dx = \int_0^1 \left( \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n \pi x \right) \left( \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos m \pi x \right) \, dx. \quad (*)$$

By orthogonality relations we have

$$\int_0^1 \cos n \pi x \cos m \pi x \, dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{1}{2} & \text{if } n = m. \end{cases}$$

Hence the right hand side of $(*)$ is equal to

$$\frac{A_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2.$$

Now observe that the left hand side of $(*)$ is equal to $\frac{1}{3}$, and that $A_0 = 1$. Hence we obtain

$$\frac{A_0^2}{4} + \sum_{n=1}^{\infty} A_n^2 = 2 \left( \frac{A_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2 \right) - \frac{A_0^2}{4} =$$

$$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$ 

2. (a) We first calculate separable solutions. It follows from the boundary conditions (fill in the details yourself!) that they are of the form $\sin nx \, e^{-n^2 t}$. It follows from the initial value conditions that the solution is given by

$$w(x,t) = \sin(2x)e^{-4t} + \sin(3x)e^{-9t}.$$ 

(b) The solution is $v(x) = x/\pi$ (check that all conditions are satisfied!)

(c) Let $\tilde{u} = u - v$, with $v$ as in (b). Then we have

$$\tilde{u}(0,t) = 0, \quad \tilde{u}(\pi,t) = 0, \quad t > 0,$$
\[ \tilde{u}(x,0) = \sin(2x) + \sin(3x). \]

Moreover, \( \tilde{u} \) also satisfies the heat equation. Hence \( \tilde{u} = w \), with \( w \) as in (a). As \( u = \tilde{u} + v \), we obtain the solution
\[ u(x,t) = \sin(2x)e^{-4t} + \sin(3x)e^{-9t} + x/\pi. \]

3. (a) Solution: \( f(r) = J_0(\sqrt{\lambda_n}r) \), where \( \lambda_n = \frac{z_0^2}{a^2} \) and \( z_0 \) is the \( n \)-th root of the derivative \( J'_0(z) \). Ask if you do not know how to justify this.

(b) Using the orthogonality of the eigenfunctions \( v_n(r) = J_0(\sqrt{\lambda_n}r) \) with respect to the inner product
\[ (f,g) = \int_0^a f(r)g(r)rdr, \]
we obtain
\[ A_n = \frac{\int_0^a r^2J_0(\sqrt{\lambda_n}r)rdr}{\int_0^a J_0^2(\sqrt{\lambda_n}r)rdr}. \]

4. (a) We use the same orthogonality arguments as in Problem 1. Hence we obtain as result
\[ (\phi,\psi) = 2(-1)\frac{1}{2} + (-4)6\frac{1}{2} + 3(0)\frac{1}{2} + (-10)(-2)\frac{1}{2} + 5(3)\frac{1}{2} + (0)(-1)\frac{1}{2} = \frac{9}{2}. \]

(b) There was a misprint in the problem: It should be \( \ldots r \) \( dr \) instead of \( dr \) at the end of the integral. The value of the integral is equal to 0 as the functions are eigenfunctions of \( -\Delta = -\nabla^2 \) for different eigenvalues, by the theorem mentioned.

5. Let us write \( u(x,y) = X(x)Y(y) \). Separating variables, we have
\[ X''(x) = -\lambda X, \quad Y''(y) = \lambda Y, \]
with boundary conditions \( X'(0) = 0 = X'((\pi)) \). These boundary conditions suggest the choice of the sign for \( \lambda \) It follows (fill in the details!) that
\[ X(x) = \cos(nx), \quad \text{with } \lambda = n^2. \]

For \( \lambda = n^2 \) the solution for \( Y \) is given by
\[ Y(y) = A_n \cosh(ny) + B_n \sinh(ny). \]

It follows from the remaining boundary conditions (justify!)\h
\[ u(x,y) = \frac{3}{\sinh 2\pi} \cos(2x)\sinh(2y) + \frac{2}{\sinh 5\pi} \cos(5x)\sinh(5y). \]