Math 18: Lectures ABC
February 28, 2018

Midterm Exam 2 v. 1
(Total Points: *30*)

Name: ______________

PID: ______________

Instructions

1. Write your Name and PID in the spaces provided above.
2. Make sure your Name is on every page.
3. No calculators, tablets, phones, or other electronic devices are allowed during this exam.
4. You may use one handwritten page of notes, but no books or other assistance during this exam.
5. Read each question carefully and answer each question completely.
6. Write your solutions clearly in the spaces provided.
7. Show all of your work. No credit will be given for unsupported answers, even if correct.

(1 point) 0. Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

(6 points) 1. Consider the matrix $A = \begin{bmatrix} 2 & 3 & 0 & 5 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$.

(a) Calculate the third column of A^{-1}.

We only need to apply the row operations which transform A to I to the third column of I which is $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 2 & 3 & 0 & 5 & 0 \\ 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 6 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} 2 & 3 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

solution: $\begin{bmatrix} -6 \\ 4 \\ -1 \\ 6 \end{bmatrix}$

(b) Let $B = \frac{1}{2}AA^{\mathsf{T}}$. Calculate $\det(B^{-1})$.

$\det A = 2 \cdot 1 \cdot (-1) \cdot 6 = -12 = \det(A^T)$

$\Rightarrow \underbrace{\det(\tfrac{1}{2} AA^T)}_{= \det B} = (\tfrac{1}{2})^4 \cdot \det A \cdot \det A^T = \frac{1}{16} \cdot 12 \cdot 12 = \frac{3}{4} \cdot 12 = 8$

$\Rightarrow \det(B^{-1}) = \frac{1}{\det B} = \frac{1}{8}$

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(9 points) 2.

The matrix $A = \begin{bmatrix} 1 & 2 & 0 & 2 & 2 \\ 2 & 4 & 1 & 9 & 3 \\ 1 & 2 & -1 & -3 & 3 \end{bmatrix}$ has reduced row-echelon form $\begin{bmatrix} 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find a basis for Nul(A).

We have free parameters x_2, x_4, x_5

and equations $x_3 + 5x_4 - x_5 = 0 \Rightarrow \boxed{x_3 = -5x_4 + x_5}$

$x_1 + 2x_2 + 2x_4 + 2x_5 = 0 \Rightarrow \boxed{x_1 = -2x_2 + 2x_4 + 2x_5}$

Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 + 2x_4 - 2x_5 \\ x_2 \\ -5x_4 + x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ basis $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) Find a basis for Col(A).

basis given by pivot columns of A = 1st and 3rd column

$\Rightarrow$ basis $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

(c) Find a basis for Col($A^{\top}$). $\cancel{=}$

basis = transpose of non-zero rows of echelon form of A

basis $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \\ -1 \end{bmatrix} \right\}$

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(8 points) 3. In each of the following examples, a vector space V is given, along with a subset S. Determine whether S is a subspace or not. In each case, explain why it is or is not a subspace.

(a) $V = M_{4\times 5}$ is the space of 4×5 matrices, and S is the set of 4×5 matrices with rank 1.

Any subspace must contain the zero vector of the vector space. Here the zero vector is the 4×5 zero matrix (all entries $=0$)
But the zero matrix has rank 0 $\Rightarrow$ not in S

$\Rightarrow$ S not a subspace

(b) $V = M_{1\times 3}$ is the space of 3-dimensional row vectors, and $S = \text{span}\{[1,1,1],[-2,2,3]\}$.

The span of any collection of vectors is a subspace

$\Rightarrow$ S is a subspace

(c) $V = \mathbb{R}^2$, and $S = \left\{ \begin{bmatrix} t \\ 3t \end{bmatrix} : 0 \le t \le 2 \right\}$.

The vector $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ is in S. For S to be a subspace it must also contain every multiple of $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$
But $3 \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$ is not in S (first coordinate t must be between 0 and 2)

(d) $V = \mathrm{P}_2$ is the space of polynomials of degree ≤ 2, and S is the subset of polynomials p in V for which $p(1) = 0$.

Let p_1 and p_2 be two polynomials in S
$(p_1+p_2)(1) = p_1(1) + p_2(1) = 0 + 0 = 0$ $\Rightarrow p_1+p_2$ in S
$cp_1(1) = c \cdot 0 = 0$ $\Rightarrow cp_1$ in S
~~He~~ The conditions for S to be a subspace are satisfied
$\Rightarrow$ S is a subspace

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(6 points) 4. Let $H = \left\{ \begin{bmatrix} s+2t \\ 2s-t \\ t \end{bmatrix} : s,t \in \mathsf{R} \right\}$. H is a subspace of R^3, with basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

(a) The vector $\mathsf{v} = \begin{bmatrix} 2 \\ 9 \\ -1 \end{bmatrix}$ is in H. Find its coordinate vector $[\mathsf{v}]_{\mathcal{B}}$.

we need to find x_1 and x_2 such that

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 2x_2 \\ 2x_1 - x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \\ -1 \end{bmatrix} \iff \begin{aligned} x_1 + 2x_2 &= 2 \\ 2x_1 - x_2 &= 9 \\ x_2 &= -1 \end{aligned}$$

$\Rightarrow \boxed{x_2 = -1}$ $\quad 2x_1 - (-1) = 9 \Rightarrow 2x_1 = 8 \Rightarrow \boxed{x_1 = 4}$

check first coord: $4 + 2(-1) = 2$ ✓

$$\Rightarrow [v]_B = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

(b) Let u be the sum of the two basis vectors in $\mathcal{B}$. Is $\{\mathsf{u}, \mathsf{v}\}$ a basis for H? Explain why or why not.

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ 9 \\ -1 \end{bmatrix}$$

H has dimension 2 (as B has 2 vectors)

$\{\vec{u}, \vec{v}\}$ are linearly independent as they are not multiples of each other

By Theorem in class (Theorem 12 in Section 4.5 in book) any 2 linearly independent vectors in a vector space of dimension 2 are a basis for that vector space.

$\Rightarrow \{\vec{u}, \vec{v}\}$ is a basis