Math 142B : Practice Midterm 2

1. True/False: Circle the correct answer. No justifications are needed in this exercise.

(1) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at 0, then $\lim_{x\to 0} \frac{f(x^2) - f(0)}{|x|} = 0$. **T / F**

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable on $\mathbb{R}$ and satisfies $f'(x) \geq 0$, for every $x \in \mathbb{R}$. Then f is strictly increasing on $\mathbb{R}$. **T / F**

(3) Let $f : (0, 2) \cup (3, 5) \to \mathbb{R}$ be a differentiable function such that $f'(x) = 0$, for every $x \in (0, 2) \cup (3, 5)$, and $f(1) = f(4)$. Then f is a constant function on $(0, 2) \cup (3, 5)$. **T / F**

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which has derivatives of all orders at every $x \in \mathbb{R}$. Assume that the Taylor series for f about 0 is identically zero. Then f is identically zero. **T / F**

(5) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$, then f' is continuous on $\mathbb{R}$. **T / F**

2. (a) Let f be a real-valued function defined on an open interval containing a point a. Define what it means for f to be differentiable at a.
(b) State the chain rule theorem.
(c) State Rolle's theorem.
(d) State the mean value theorem.
(e) State the intermediate value theorem for derivatives.
(f) Let f be a real-valued function defined on an interval I. Define what it means for f to be strictly decreasing on I.
(g) Let f be a real-valued function defined on an open interval containing a point c. Assume that f has derivatives of all orders at c. Define the Taylor series for f about c.

3. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2, \text{ if } x \text{ is rational} \\ x^3, \text{ if } x \text{ is irrational} \end{cases}$
Prove that f is differentiable at 0 and $f'(0) = 0$.
(b) Show that $\lim_{x\to\infty} \sqrt{x} \sin(\frac{1}{x}) = 0$.

4. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions.
(a) Assume that $f'(x) < 1$, for every $x \in \mathbb{R}$. Prove that the equation $f(x) = x$ has at most one solution $x \in \mathbb{R}$.
(b) Assume that $|g(x) - g(y)| \geq |x - y|$, for every $x, y \in \mathbb{R}$. Prove that $|g'(x)| \geq 1$, for every $x \in \mathbb{R}$. Deduce that either $g'(x) \geq 1$, for every $x \in \mathbb{R}$, or $g'(x) \leq -1$, for every $x \in \mathbb{R}$.

5. (a) Use Taylor's theorem to prove that $1 + \frac{3}{2}x < (1 + x)^{\frac{3}{2}} < 1 + \frac{3}{2}x + \frac{3}{8}x^2$, for all $x > 0$.
(b) Find the Taylor series for the function $f(x) = e^x - e^{-x} - 2\sin x$ about 0 and show that it converges to $f(x)$, for every $x \in \mathbb{R}$.
(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which has derivatives of all orders such that $|f^{(n)}(x)| \leq n!$, for any $x \in (-1, 1)$ and $n \in \mathbb{N}$. Prove that $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$, for every $x \in (-1, 1)$.