Math 142B : Additional practice problems

1. True/False: Circle the correct answer. No justifications are needed in this exercise.

(1) If $f : [0,1] \to \mathbb{R}$ is integrable on $[0,1]$, then f is continuous on $[0,1]$. **T / F**

(2) Let (f_n) be a sequence of integrable functions on $[0,1]$. Assume that $\lim\limits_{n\to\infty} f_n(x) = 0$, for every $x \in [0,1]$. Then $\lim_{n\to\infty} \int_0^1 f_n = 0$. **T / F**

(3) If f is a continuous function on $\mathbb{R}$, then there exists a function F which is differentiable on $\mathbb{R}$ such that $f = F'$. **T / F**

2. (a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Let $P = \{a = t_0 < t_1 < ... < t_n = b\}$ be a partition of $[a,b]$. Give the definition of the upper Darboux sum $U(f,P)$ of f with respect to P. Give the defintion of a Riemann sum of f associated with P.

(b) State the Intermediate Value Theorem for Integrals.

(c) State what it means for a function $f : [a,b] \to \mathbb{R}$ to be piecewise continuous.

3. Let $f : [0,1] \to \mathbb{R}$ be a function defined by $f(x) = \begin{cases} 0, & \text{if } x \in [0,1] \text{ is rational} \\ 2x, & \text{if } x \in [0,1] \text{ is irrational} \end{cases}$

Let $n \geq 1$ and consider the partition $P_n = \{0 = t_0 < t_1 < ... < t_n = 1\}$, where $t_k = \frac{k}{n}$.

(a) Calculate $L(f, P_n)$.

(b) Calculate $U(f, P_n)$.

(c) Prove that f is not integrable on $[0,1]$.

4. Let $f : [a,b] \to \mathbb{R}$ be an increasing function.

Prove that $U(f,P) - L(f,P) \leq \text{mesh}(P) \cdot [f(b) - f(a)]$, for every partition P of $[a,b]$.

5. Let f and g be continuous functions on $[0,1]$. Assume that $g(x) > 0$, for every $x \in [0,1]$.

Prove that there exists $x \in (0,1)$ such that $\dfrac{f(x)}{g(x)} = \dfrac{\int_0^1 f}{\int_0^1 g}$.

6. Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $g(\mathbb{R}) = \mathbb{R}$ and $g'(x) > 0$, for all $x \in \mathbb{R}$. Let $G : \mathbb{R} \to \mathbb{R}$ be given by $G(x) = \int_0^{g(x)} g^{-1}(t)\, dt$, for every $x \in \mathbb{R}$.

Prove that G is differentiable on $\mathbb{R}$ and $G'(x) = xg'(x)$, for every $x \in \mathbb{R}$.

Hint. Use the fact that $G(x) = F(g(x))$, where $F(x) = \int_0^x g^{-1}(t)\, dt$.