MATH 103 FINAL FALL 2009

Please justify all your steps!

1. Calculate the order of $(10, 12)$ in $\mathbf{Z}_{12} \oplus \mathbf{Z}_{16}$.

2. (a) Fix a positive integer $n \in \mathbf{N}$. Show that the map $x \mapsto nx \bmod 10$ is a homomorphism from $\mathbf{Z}_{10}$ into itself.
(b) Determine those n for which the map in (a) is an isomorphism ($Hint$: You only need to consider $0 \leq n < 10$).

3. (a) Is the following true: For every positive integer $n \in \mathbf{N}$ there exists a group G of order n. Either give examples for all n, or produce an n for which there is no group with $|G| = n$.
(b) Every group of order p, with p a prime number, is abelian. Why or why not?
(c) Every group of order $2p$, with p a prime number, is abelian. Why or why not?
(d) There exists a group with 44 elements which contains an element of order 8. Why or why not?

4. Let G be the set of all 2×2 matrices A with *integer* entries such that the determinant of A is equal to 1. Also recall the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(a) Show that G is a group (you may assume that matrix multiplication is associative).
(b) Fix $n \in \mathbf{N}$. Let $H \subset G$ be the set of matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a \equiv d \equiv 1 \bmod n$, and $c \equiv d \equiv 0 \bmod n$. Show that H is a subgroup of G.

5. Let $G = S_6$, the group of all permutations of the integers 1 until 6. Let H be the subgroup of G of all permutations σ with $\sigma(1) = 1$. Moreover, let $\pi = (132)$.
(a) Show for the left coset πH that $\pi H = \{\gamma \in S_n, \gamma(1) = 3\}$.
(b) If β is an element in the *right* coset $H\pi$, what is $\beta(2)$?
(c) Is H a normal subgroup of S_6?

6. Let $G = H \oplus K$ be the external direct product of the groups H and K. Consider the map $\Phi : G \to H$ which maps (h, k) to h.
(a) Show that Φ is a homomorphism.
(b) What is the kernel of Φ?

7. Let $G = \mathbf{Z}_4 \oplus \mathbf{Z}_2$ and let $a = (2, 1) \in G$.
(a) What is the order of the factor group $|G/\langle a\rangle|$?
(b) To which group is $G/\langle a\rangle$ isomorphic?

8. Prove or disprove that $U(8)$ is isomorphic to $U(5)$.