Useful calculation:
\[ (\sin \frac{m\pi}{L} x, \sin \frac{n\pi}{L} x) = \int_0^L \sin \frac{m\pi}{L} x \sin \frac{n\pi}{L} x \, dx \]

This can be solved using trig identity:
\[ \sin x \sin y = \frac{1}{2} (\cos(x-y) - \cos(x+y)) \]

You can derive this for yourself using:
\[ \sin x = \frac{1}{2i} (e^{ix} - e^{-ix}), \quad \cos x = \frac{1}{2} (e^{ix} + e^{-ix}) \]

\[ = \frac{1}{2} \int_0^L \cos \frac{(m-n)\pi}{L} x - \cos \frac{(m+n)\pi}{L} x \, dx \]

\[ = \frac{1}{2} \left[ \frac{L}{(m-n)\pi} \sin \frac{(m-n)\pi}{L} x - \frac{L}{(n+m)\pi} \sin \frac{(n+m)\pi}{L} x \right]_0^L \]

\[ = \frac{1}{2} \left[ \frac{L}{(m-n)\pi} \sin \frac{(m-n)\pi}{L} L - \frac{L}{(n+m)\pi} \sin \frac{(n+m)\pi}{L} L \right] \]

\[ \text{except if } n = m \text{ by calculations on next page} \]
Last class:
\[
\int_{0}^{L} \sin \frac{m\pi}{L} x \sin \frac{n\pi}{L} x \, dx = \begin{cases} 
L/2 & \text{if } n=m \\
0 & \text{if } n \neq m
\end{cases}
\]

Recall: we defined \( (f, g) = \int_{0}^{L} f(x)g(x) \, dx \)

\[
\Rightarrow (\sin \frac{m\pi}{L} x, \sin \frac{n\pi}{L} x) = \begin{cases} 
L/2 & \text{if } n=m \\
0 & \text{if } n \neq m
\end{cases}
\]

Recall from linear algebra:
If \( a_1, a_2, \ldots, a_d \) in \( \mathbb{R}^d \) nonzero vectors
s.t. \( (u_n, u_m) = u_n \cdot u_m = 0 \) for \( n \neq m \)
and \( v \in \mathbb{R}^d \)

\[
\Rightarrow v = \sum_{n=1}^{d} b_n u_n \quad \text{where} \quad b_n = \frac{(v, u_n)}{(u_n, u_n)}
\]
Same formula works for infinite dim. vector spaces here: \( V = \) all integrable functions on interval \([0, L]\)

\[ u_n = \sin \frac{n\pi}{L} x \]

**Theorem:** Let \( f \) be continuous function on \([0, L]\), \( f(0) = f(L) \).

\[ f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x \]

where \( B_n = \frac{2}{L} \int f(x) \sin \frac{n\pi}{L} x \, dx \)

\[ \frac{2}{L} = \langle u_n, u_n \rangle \]

Remark: theorem also works for not necessarily continuous function.
This was last missing piece for solving

(PDE) \[ \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \]

(BC) \[ u(0, t) = 0 = u(L, t) \]

(IC) \[ u(x, 0) = f(x) \quad \text{given function } f \]
Strategy to solve:

1. Calculate product solutions \( u(x,t) = G(t)\Phi(x) \)

2. Separate variables

\[
\frac{G'(t)}{kG(t)} = \frac{\Phi''(x)}{\Phi(x)} = -\lambda
\]

3. Solve two ODE's

\[ G'(t) = -\lambda k G(t) \]

and \[ \Phi''(x) = -\lambda \Phi(x) \]

4. Use boundary conditions to determine possible values of \( \lambda \) (in our example \( \lambda = \frac{n^2\pi^2}{L^2} \))

Def. \( \lambda \) is called an eigenvalue of our PDE with given boundary conditions.
For each eigenvalue $\lambda$ find corresponding solution $\phi_\lambda(x)$ of ODE $\phi''(x) = -\lambda \phi(x)$

(in our example: $\lambda = \frac{n^2 \pi^2}{L^2}$

$\phi_\lambda = \sin \frac{n \pi}{L} x$)

(b) Find Fourier expansion of $f(x)$ in terms of eigenfunctions $\phi_\lambda$

(in our example: $f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n \pi}{L} x$

where $B_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n \pi}{L} x dx$)

(c) Solution: $u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n \pi}{L} x e^{-(\frac{n \pi}{L})^2 \alpha t}$
Concrete example:
\[ f(x) = 100 \quad \text{for} \quad 0 < x < 100 \]
\[ f(0) = 0 = f(100) \]

Solution: have already shown:

\[ \text{can expand} \quad f(x) = \sum B_n \sin \frac{n\pi x}{L} \]

\[ \Rightarrow \text{get solution} \]
\[ u(x, t) = \sum B_n \sin \frac{n\pi x}{L} \times e^{-\left(\frac{(n\pi)^2}{L^2}\right) t} \]

only thing left: calculate coefficients \( B_n \)
\[ B_n = \frac{2}{L} \int_{0}^{L} 100 \cdot \sin \frac{m \pi x}{L} \cdot f(x) \, dx \]

\[ = \frac{200}{m \pi} \left[ -\cos \frac{m \pi x}{L} \right]_{0}^{L} \]

\[ = \frac{200}{m \pi} \left( -\cos \frac{m \pi L}{L} + \cos \frac{m \pi 0}{L} \right) \]

\[ = \frac{200}{m \pi} \left( 1 - \cos \frac{m \pi L}{L} \right) = \frac{200}{m \pi} \left( 1 - (-1)^n \right) \]

\[ = \left\{ \begin{array}{ll}
\frac{400}{m \pi} & n \text{ odd} \\
0 & n \text{ even}
\end{array} \right. \]
\( u(x,t) \approx \frac{400}{\pi} \sin \left( \frac{\pi}{L} x \right) e^{-\left( \frac{\pi}{L} \right)^2 \pi t} \)

See picture in book.

Max. amplitude at time \( t = \frac{\pi}{11} e^{-\left( \frac{\pi}{L} \right)^2 \pi t} \) reached at \( x = \frac{L}{2} \)

Precise description of cooling off of a rod with initial temperature 100\(^\circ\) with constant boundary temp 0\(^\circ\).