Last class: Derivation of Wave Equation

Set-up:

\[ u \]

\[ \text{only vertical movement} \]

(say string in a guitar)

calculate \( F = ma \) Newton's Law

in 2 different ways for a small segment of string

1. \( m \approx g(x) \Delta x \)

\( g(x) = \text{mass density/length unit} \)

\( a = \frac{\partial^2 u}{\partial t^2} (x,t) \)

Precise value: \( \int_{x}^{x+\Delta x} g(y) \, dy \)

\[ F = g(x) \Delta x \frac{\partial^2 u}{\partial t^2} (x,t) \]
2) Study forces explicitly

- forces coming from tension on string

\[ T(x, t) \]

\[ T(x+\Delta x, t) \]

**Assumptions:**
- String elastic = tension forces
- Movement only in vertical direction
- \( T(x+\Delta x, t) \sin(\theta(x+\Delta x, t)) \)

\[ \theta \text{ small} \]

\[ \sin \theta = \frac{a}{c} \]
\[ \tan \theta = \frac{a}{b} \]
\[ c \approx b \text{ for small } \theta \]

\[ \tan(\theta(\ldots)) = u(x+\Delta x, t) = \frac{\partial x u(x+\Delta x, t)}{\partial x} \]
\[ T(x+\Delta x, t) \sin \theta(x+\Delta x, t) \]
\[ = T(x+\Delta x, t) \frac{\partial u}{\partial x}(x+\Delta x, t) \]

Same for \( T(x, t) \sin \theta(x, t) \)
\[ = \frac{\partial u}{\partial x}(x, t) \]

\[ \Rightarrow \text{force coming from tension of string} \]

\[ = T(x+\Delta x, t) \frac{\partial u}{\partial x}(x+\Delta x, t) - T(x, t) \frac{\partial u}{\partial x}(x, t) \]

Pull in different directions

- Additional forces:
  given by \( \alpha(x, t) g(x) \Delta x \)

  Usually: \( \alpha(x, t) = -g = \text{gravity} \)

\[ \alpha(x, t) = \text{force/mass unit} \]
\[
\sigma(x) \Delta x \frac{\partial^2 u}{\partial t^2} (x,t) = ma = F = \\
T(x + \Delta x, t) \frac{\partial u}{\partial x} (x + \Delta x, t) - T(x, t) \frac{\partial u}{\partial x} (x, t) \\
+ \delta(x) \Delta x \alpha(x, t)
\]

\[
\Rightarrow \sigma(x) \frac{\partial^2 u}{\partial t^2} (x,t) = \frac{1}{\Delta x} \left( \left[T(x + \Delta x, t) \frac{\partial u}{\partial x} (x + \Delta x, t) - T(x, t) \frac{\partial u}{\partial x} (x, t) \right] \\
+ \delta(x) \alpha(x,t) \right)
\]

If \( \Delta x \to 0 \) get

\[
\sigma(x) \frac{\partial^2 u}{\partial t^2} = \partial_x \left( T(x,t) \frac{\partial u}{\partial x} (x,t) \right) + \delta(x) \alpha(x,t)
\]
Simplifications:

Assume \( s \) and \( T \) to be constant.

\[
\begin{align*}
  s \frac{\partial^2 u}{\partial t^2} &= T \frac{\partial^2 u}{\partial x^2} + s \alpha(x,t) \\
  \text{equilibrium state:} \quad \frac{\partial^2 u}{\partial t^2} &= 0 \\
  \Rightarrow \quad & \text{get ODE} \\
  T \frac{\partial^2 u}{\partial x^2} + s \alpha(x,t) &= 0
\end{align*}
\]

Often \( \alpha \) is negligible compared to \( T \)

\[
\begin{align*}
  s \frac{\partial^2 u}{\partial t^2} &= T \frac{\partial^2 u}{\partial x^2} \\
  \Rightarrow \quad & \text{one-dimensional wave equation} \\
  \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where} \quad c^2 = \frac{T}{s}
\end{align*}
\]
4.4 Wave equation with Fixed boundaries

Problem:

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]  

(POE)

\[ u(0,t) = 0 = u(L,t) \]  

(BC)

\[ u(x,0) = f(x) \]  

(I.C)

\[ \frac{\partial u}{\partial t}(x,0) = g(x) \]

Remark: have second order DE with respect to \( t \)  

\( \Rightarrow \) need two initial conditions
We can solve this by some methods as for heat equation!

- Consider product solutions $u(x, t) = \phi(x) h(t)$
- Plug into PDE $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

\[ \Rightarrow \quad \phi(x) h''(t) = c^2 \phi''(x) h(t) \]

\[ \Rightarrow \quad \frac{h''(t)}{c^2 h(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda \]

\((BC): u(0, t) = 0 = u(L, t) \Rightarrow \phi(0) = 0 = \phi(L) \]

\[ \phi''(x) = -\lambda \phi(x) \]
as before get:

\[ \phi(x) = \sin \left( \frac{n\pi x}{L} \right) \]  
\[ \lambda = \left( \frac{n\pi}{L} \right)^2 \]

Similarly:
\[ h'' = -\lambda c^2 h \]
\[ h = -\left( \frac{n\pi c}{L} \right)^2 h \]  
\[ h(x) = A_n \cos \frac{n\pi c}{L} t + B_n \sin \frac{n\pi c}{L} t \]

⇒ get series solution,
\[ u(x,t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi c}{L} t \sin \frac{n\pi x}{L} x + B_n \sin \frac{n\pi c}{L} t \sin \frac{n\pi x}{L} x \]
Calculate Fourier coefficients using initial conditions

\[ u(x, 0) = f(x) \]
\[ \Rightarrow \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = f(x) \]
\[ (\cos \frac{n\pi x}{L})_0 = 1 \]

\[ A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \]

\[ \frac{\partial u}{\partial t} (x, 0) = g(x) \]
\[ \Rightarrow \sum_{n=1}^{\infty} B_n \frac{n\pi x}{L} \sin \frac{n\pi x}{L} = g(x) \]
\[ B_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx \]