Last class:

- Found out that differentiating sine series for \( x = f(t) \) → got a non-converging cosine series ≠ cosine series of \( \frac{d}{dx}(x) = 1 \)

Clarification for this fact:

**Theorem:** \( f: [a, b] \rightarrow \mathbb{R} \) (\( f \) continuous) ⇒ \( f'(x) \) piecewise smooth ⇒ \( f(x) \) has Fourier series which converges.

- Differentiating sine series for \( f \) term by term ⇒ get cosine series for \( f' \) only if \( f(a) = f(b) = 0 \)
- Differentiating cosine series for \( f \) term by term ⇒ get sine series for \( f' \)
Proof for \( a \)

given sine series of \( f \)

\[ f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \]

Differentiate sine series term by term:

\[ \Rightarrow \sum_{n=1}^{\infty} B_n \frac{n\pi}{L} \cos \frac{n\pi x}{L} \]  

Let \( f'(x) \sim \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} \)  

Cosine series for \( f' \)

Question: Are these two series the same?

\[ (\Rightarrow) \]

\[ A_0 = 0 \]

\[ A_n = \frac{B_n n\pi}{L} \quad n > 0 \]
Calculate the \( A_n \)'s!

\[
A_0 = \frac{2}{L} \int_0^L f'(x) \, dx
\]

\[
= \frac{2}{L} \left( f(L) - f(0) \right) = 0
\]

\( n > 0 \)

\[
A_n = \frac{1}{L} \int_0^L f'(x) \cos \frac{n\pi x}{L} \, dx
\]

\[
= \frac{1}{L} \left[ f(x) \cos \frac{n\pi x}{L} \bigg|_0^L + \int_0^L f(x) \left( -\frac{n\pi}{L} \sin \frac{n\pi x}{L} \right) \, dx \right]
\]

int. by parts

\[
= \frac{1}{L} \left[ f(L) \cos \frac{n\pi L}{L} - f(0) + \int_0^L f(x) \frac{n\pi}{L} \sin \frac{n\pi x}{L} \, dx \right]
\]

\[
= \frac{1}{L} \left[ f(L) \cos \frac{n\pi L}{L} - f(0) + (-1)^m \int_0^L f(x) \frac{n\pi}{L} \sin \frac{n\pi x}{L} \, dx \right]
\]
\[ \frac{1}{L} \left[ (-1)^{n+1} f(L) - f(0) \right] + \frac{\pi n}{L} \int_0^L f(x) \sin \frac{\pi n x}{L} \, dx \]

\[ \frac{1}{L} \left[ (-1)^{n+1} f(L) - f(0) \right] + \frac{\pi n}{L} B_n \]

\text{Result: Coefficient } A_n \text{ of cosine series of } f' \]

\[ A_n = \frac{\pi n}{L} B_n \quad \Rightarrow \quad f(L) = 0 = f(0) \]

\[ (-1)^n f(L) - f(0) = 0 \text{ both for } n \text{ odd} \]

and for \( n \) even

\[ (-1)^n f(L) - f(0) = 0 \]

\text{bad news: need to be careful if } f(0) \text{ to } a \text{ or } f(L) \to 0 \]

\text{good news: even if } f(0) \to a \text{ or } f(L) \to 0 \text{ we can calculate cosine series of } f' \text{ from sine series of } f \text{ via here: } f \text{ needs to be continuous on } [0, L] \]
Remark: Integrating Fourier series term by term is less complicated.

If $F$ is an antiderivative of $f$, we can obtain its Fourier series from the one of $F$ always by integrating term by term up to a constant.
4. Wave Equation

1-dim wave equation.

physical set-up.

string spanned between

assumptions:

- string segments only move in vertical direction
- \( u(x,t) \) = position of string segment at \( x \) at time \( t \) in vertical

- \( \rho_0(x) \) = mass density at \( x \) (usually constant)

- Newton's law: \( F = ma \) (force = mass \( \times \) acceleration)
- string perfectly flexible
  - tension forces always going in direction of tangent

\[ u \]

\[ x \quad x + \Delta x \]

vertical component

\[ \sin \theta T(x,t) \]

relevant part of tension force

\[ \sin \theta = \text{slope at } x = \frac{\partial u}{\partial x}(x,t) \]

determine force on line segment between \( x \) and \( x + \Delta x \)

\[ T(x,t) \frac{\partial u}{\partial x}(x,t) \]
\[ F = m \alpha \]

\[ = s_0(x) \Delta x \frac{\partial^2 u}{\partial t^2} \]

\[ = m \]

\[ = \alpha \]

= forces from tension + exterior forces (e.g., gravity)

act on endpoints of line segment

\[ = \frac{\partial u}{\partial x}(x+\Delta x,t) \Pi(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t) \Pi(x,t) \]

\[ + s_0(x) \Delta x \frac{\partial^2 u}{\partial x^2} \quad \text{mass} \quad \text{exterior force} \]

\[ \text{get} \]

\[ s_0(x) \Delta x \frac{\partial^2 u}{\partial t^2} = \]