MATH 110
FINAL

Please answer the following questions.You *will not get partial credit* for answers unless you demonstrate how you arrived at them. In short, please show all work.

Problem 1.

Please compute the specific solution to the transport equation in the region where $y \geqslant 1$:

$$\begin{aligned} e^x \partial_x u + y \partial_y u &= 0 \ , \\ u(x,1) &= e^{-x} \ . \end{aligned}$$

Problem 2.

Consider the solution $u(x,t)$ to the wave equation:

$$\begin{aligned} \partial_t^2 u &= \partial_x^2 u \ , \\ u(x,0) &= \begin{cases} \cos(x) \ , & \text{if } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \ ; \\ 0 \ , & \text{otherwise .} \end{cases} \\ u_t(x,0) &= \begin{cases} \sin(x) \ , & \text{if } -\pi \leqslant x \leqslant \pi \ ; \\ 0 \ , & \text{otherwise .} \end{cases} \end{aligned}$$

Please compute the quantity $u(10,10)$.

Problem 3.

Please compute the energy:

$$E[u](t) \;=\; \frac{1}{2}\int_{-\infty}^{\infty} \big(u_t^2 + u_x^2\big)(x,t)\; dx \;,$$

of the solution $u(x,t)$ to the previous problem at time $t = 10$.

Problem 4.

The heat equation for a solid metal ring is the following ("periodic" boundary conditions):

$$\begin{aligned} u_t &= k u_{xx} \ , & -\pi \leqslant x \leqslant \pi \ , \\ u(-\pi, t) &= u(\pi, t) \ , \\ u_x(-\pi, t) &= u_x(\pi, t) \ , \\ u(x, 0) &= f(x) \ . \end{aligned}$$

a) Use separation of variables to derive the general solution $u(x,t)$ to this problem.

b) Write down the specific solution $u(x,t)$ to the above problem with initial data:

$$f(x) \ = \ 3 + 2\cos(3x) - 5\sin(4x) \ .$$

c) Compute the steady state temperature $T_\infty = \lim_{t\to\infty} u(x,t)$ to the solution from part b). Show that this is equal to the average initial temerature:

$$T_{av} \ = \ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx \ .$$

Problem 5.

Please compute the full sin/cos series on the interval $[-1, 1]$ for the function:

$$\phi(x) \;=\; 1 - x^2 \;.$$

(Hint: Notice that this function is even. Also, use integration by parts to compute the integrals.)

Problem 6.

a) Solve the following Dirichlet problem in the unit circle:

$$\begin{aligned} \Delta u &= 0 \,, && \text{in } \ 0 \leqslant r < 1 \,, \\ u(1,\theta) &= 1 + 3\sin(5\theta) + \cos(6\theta) - 4\cos(10\theta) \,. \end{aligned}$$

b) Compute the average value:

$$A = \frac{1}{2\pi} \int_0^{2\pi} u(1,\theta)\, d\theta \,,$$

and show that it is equal to $u|_{r=0}$.