BESSEL FUNCTIONS

Here we collect a number of useful properties of Bessel functions. Using its power series expansion

$$J_s(z) = \sum_{j=0}^{\infty} \frac{1}{\Gamma(j+1)\Gamma(j+s+1)} \, (\frac{z}{2})^{s+2j},$$

one can prove (see homework) that

$$J_{s\pm 1}(z) = \frac{s}{z} J_s(z) \mp J_s'(z). \tag{1}$$

We have shown the following result in class.

Lemma 1 With the definitions above, we have

(*a*) $\int_0^a J_s(z)^2 z \, dz = \frac{1}{2}[a^2 J_s'(a)^2 + (a^2 - s^2) J_s(a)^2]$,

(b) If $J_s(\beta a) = 0$, then

$$\int_0^a J_s(\beta r)^2 r \, dr = \frac{1}{2} a^2 J_s'(\beta a)^2 = a^2 J_{s\pm 1}(\beta a)^2,$$

(*c*) $(\tilde{u}_{n,m}^{(\nu)}, \tilde{u}_{n,m}^{(\nu)}) \;=\; \frac{\pi}{2} a^2 J_{s\pm 1}(\sqrt{\lambda_{n,m}} a)^2.$

Proof. For the proofs of (a) and (b), see p 284, *Normalizing Constants* in Chapter 10.5 of our book. For part (c) we use the definition of $\tilde{u}_{n,m}^{(\nu)}$ and of the inner product $(\tilde{u}_{n,m}^{(\nu)}, \tilde{u}_{n,m}^{(\nu)})$, see the proof of the first theorem in the previous notes on vibrations of a drumhead. It then suffices to apply part (b) to the integral, with $\beta = \sqrt{\lambda_{n,m}}$.

We shall also need Bessel functions J_s with $s = n + \frac{1}{2}$ a half-integer. Using the substitution $u = z^{-1/2} v$, it is shown in a homework problem that Bessel's equation transforms to

$$v'' + (1 - \frac{s^2 - \frac{1}{4}}{z^2}) v = 0.$$

It follows that for $s = \frac{1}{2}$ we get the differential equation $v'' + v = 0$. As we also want a finite value for $u(z) = z^{-1/2} v(z)$ for $z \to 0$, we obtain the solution

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z,$$

where the factor $\sqrt{\frac{2}{\pi}}$ is added for making certain other formulas nicer later on.

Lemma 2 The half-integer Bessel functions $J_{n+\frac{1}{2}}$ are given by the formula

$$J_{n+\frac{1}{2}}(z) = (-1)^n \sqrt{\frac{2}{\pi}} \; z^{n+\frac{1}{2}} \left(z^{-1} \frac{d}{dz} \right)^n \frac{\sin z}{z}.$$

Proof. The proof goes by induction on n. It is easily checked for $n = 0$. By induction assumption for $n - 1$, we can write

$$\left(z^{-1} \frac{d}{dz} \right)^{n-1} \frac{\sin z}{z} = (-1)^{n-1} z^{\frac{1}{2} - n} \sqrt{\frac{\pi}{2}} J_{n-\frac{1}{2}}(z).$$

Using this, we can reduce differentiating n times to only differentiating once in the formula below:

$$(-1)^n\sqrt{\frac{2}{\pi}}\; z^{n+\frac{1}{2}}\left(z^{-1}\frac{d}{dz}\right)^n\frac{\sin z}{z} \;=\; -z^{n-\frac{1}{2}}\frac{d}{dz}(J_{n-\frac{1}{2}}(z)z^{\frac{1}{2}-n}) \;=$$

$$=\; -z^{n-\frac{1}{2}}[J'_{n-\frac{1}{2}}(z)z^{\frac{1}{2}-n}+(\frac{1}{2}-n)z^{-\frac{1}{2}-n}J_{n-\frac{1}{2}}(z)\;] \;=$$

$$=\; -[J'_{n-\frac{1}{2}}(z)-\frac{n-\frac{1}{2}}{z}J_{n-\frac{1}{2}}(z)] \;=\; J_{n+\frac{1}{2}}(z),$$

where we used the recursive formula (1) for $s = n - \frac{1}{2}$. This finishes the proof.

Example Check that I did not make a mistake by calculating

$$J_{3/2} = \sqrt{\frac{2}{\pi}}z^{-3/2}(\sin z - z\cos z).$$

What about its limit if $z \to 0$? Recall that it should be finite!