MATH 210B WINTER 2001: MIDTERM

1. Let V be the vector space of polynomials of degree ≤ 2 on the interval [-1,1], and let $\langle\ ,\ \rangle$ be the usual inner product, given by $\langle f, g\rangle = \int_{-1}^{1} \bar{f}(x)g(x)dx$.
   a) Prove one of the following identities (whichever you like): $\langle 1, 1\rangle = 2, \langle x, x\rangle = 2/3 =$
   $= \langle 1, x^2\rangle, \langle x^2, x^2\rangle = 2/5$ and $\langle x^i, x^j\rangle = 0$ whenever $i - j$ odd (for $0 \leq i, j \leq 2$).
   b) Let $D : V \to V$ be defined by $Df = f'$. Compute $D^\dagger x^2$. (hint: Write $D^\dagger x^2 = ax^2 + bx + c$ and find equations for the coefficients a, b, c. You may use all the identities in a).

2. Let V be an inner product space and let $A : V \to V$ be linear. Show that all eigenvalues of $A^\dagger A$ are real and positive.

3. Let $H : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ be the operator related to the harmonic oscillator, defined by $H(f) = x^2 f - f''$. Moreover, define for any operator $A : V \to V$ the operator $exp(A)$ by its power series $exp(A) = \sum \frac{1}{m!} A^m$. Show that $exp(iH)$ is a unitary operator (hint: you may use that $L^2(\mathbf{R})$ contains a complete orthogonal set $\{H_n e^{-x^2/2}, n \in \mathbf{N}\}$, where H_n is the n-th Hermite polynomial, and that these are eigenvectors of H).