

1. Review sheet

A. The four types of integrals

1. Integral of scalar function f over curve γ

Definition by parametrising $\mathbf{s} : [t_0, t_1] \rightarrow \gamma$:
$$\int_{\gamma} f \, ds = \int_{t_0}^{t_1} f(\mathbf{s}(t)) \cdot \left\| \frac{d\mathbf{s}}{dt} \right\| dt$$

Relation to arclength:
$$\int_{\gamma} 1 \, ds = \text{length}(\gamma)$$

Relation to average:
$$\int_{\gamma} f \, ds = \text{average}_{\gamma}(f) \times \text{length}(\gamma)$$

2. Integral of scalar function f over surface Σ

Definition by parametrising $\Phi : D \rightarrow \Sigma$:
$$\int_{\Sigma} f \, dA = \iint_D f(\Phi(u, v)) \cdot \|\mathbf{T}_u \times \mathbf{T}_v\| \, du \, dv$$

Relation to surface area:
$$\int_{\Sigma} 1 \, dA = \text{area}(\Sigma)$$

Relation to average:
$$\int_{\Sigma} f \, dA = \text{average}_{\Sigma}(f) \times \text{area}(\Sigma)$$

3. Integral of vector field \mathbf{v} along oriented curve γ (called “circulation” if γ is closed)

Definition by parametrising $\mathbf{s} : [t_0, t_1] \rightarrow \gamma$:
$$\int_{\gamma} \mathbf{v} \cdot d\mathbf{s} = \int_{t_0}^{t_1} \mathbf{v}(\mathbf{s}(t)) \cdot \frac{d\mathbf{s}}{dt} dt$$

Expression in terms of scalar integral:
$$\int_{\gamma} \mathbf{v} \cdot d\mathbf{s} = \int_{\gamma} (\mathbf{v} \cdot \hat{\mathbf{t}}) \, ds$$

Relation to average:
$$\int_{\gamma} \mathbf{v} \cdot d\mathbf{s} = \text{average}_{\gamma}(\mathbf{v} \cdot \hat{\mathbf{t}}) \times \text{length}(\gamma)$$

4. Integral of vector field \mathbf{v} across an oriented surface Σ (called “flux”)

Definition by parametrising $\Phi : D \rightarrow \Sigma$:
$$\int_{\Sigma} \mathbf{v} \cdot d\mathbf{A} = \iint_D \mathbf{v}(\Phi(u, v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) \, du \, dv$$

Expression in terms of scalar integral:
$$\int_{\Sigma} \mathbf{v} \cdot d\mathbf{A} = \int_{\Sigma} (\mathbf{v} \cdot \hat{\mathbf{n}}) \, dA$$

Relation to average:
$$\int_{\Sigma} \mathbf{v} \cdot d\mathbf{A} = \text{average}_{\Sigma}(\mathbf{v} \cdot \hat{\mathbf{n}}) \times \text{area}(\Sigma)$$

B. General advice on evaluating integrals

1. Start by identifying which of the four types of integral (curve or surface; scalar or vector) you are asked to compute! (I'll just talk about the two types of surface integral below, since the two types of curve integral are very similar, but easier.)

$$\int_{\gamma} f \, ds \quad \int_{\Sigma} f \, dA \quad \int_{\gamma} \mathbf{v} \cdot d\mathbf{s} \quad \int_{\Sigma} \mathbf{v} \cdot d\mathbf{A}$$

2. The “default” method of evaluation is straightforward. First choose a parametrisation $\Phi(u, v)$ of the surface Σ in terms of some domain D in the uv -plane, where u and v are your new variables. Now use the “defining” formula (the one with $\mathbf{T}_u \times \mathbf{T}_v$ or $\|\mathbf{T}_u \times \mathbf{T}_v\|$ in it, according to whether we have a vector or scalar integral) to write it down as a double integral over D “ $du \, dv$ ”.

Important: be careful to distinguish during your calculations whether your integrals are over the original surface Σ or the parametrisation domain D : they are completely different things!

3. Some standard types of parametrisation are as follows:

(a). If the surface is the graph of some function $z = g(x, y)$, we can just take the parametrisation to be $\Phi(u, v) = (u, v, g(u, v))$ and then speed things up using the formulae

$$\mathbf{T}_u \times \mathbf{T}_v = (-g_u, -g_v, 1) \quad \text{or} \quad \|\mathbf{T}_u \times \mathbf{T}_v\| = \sqrt{1 + g_u^2 + g_v^2}$$

(b). For part of a sphere of radius a we can use the standard spherical coordinates given by $\Phi(\theta, \phi) = (a \cos \theta \sin \phi, a \sin \theta \sin \phi, a \cos \phi)$ and use the standard formulae

$$\mathbf{T}_{\theta} \times \mathbf{T}_{\phi} = -a^2 \sin \phi (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \quad \text{or} \quad \|\mathbf{T}_{\theta} \times \mathbf{T}_{\phi}\| = a^2 \sin \phi$$

(c). For part of a vertical cylinder of radius a , we can use the standard cylindrical coordinates given by $\Phi(\theta, z) = (a \cos \theta, a \sin \theta, z)$ and use the formulae

$$\mathbf{T}_{\theta} \times \mathbf{T}_z = (a \cos \theta, a \sin \theta, 0) \quad \text{or} \quad \|\mathbf{T}_{\theta} \times \mathbf{T}_z\| = a$$

C. Useful tricks for evaluating integrals

There's nothing wrong with the default method given above – it should always work as long as the functions involved in the integrand and the parametrisation aren't too complicated – but it's not always the fastest method. We can rewrite integrals in various ways, and *sometimes* (but not always!) one of these alternatives turns out to be easier to actually compute, so it's definitely worth being aware of these possibilities.

1. The nicest kind of integral is one you can calculate without integrating, such as $\int_{\Sigma} 5 \, dA = 5 \text{ area}(\Sigma)$ or $\iint_D 5 \, dudv = 5 \text{ area}(D)$. If you can calculate the area just using geometry, it's usually a lot quicker than using calculus! Notice that it's critical here that you know whether you're integrating over the actual surface Σ or the parametrisation domain D (see note 2 above).

2. Symmetry tricks. The simplest is just that any integral over the sphere of the form

$$I = \int_{\Sigma} x^p y^q z^r \, dA$$

where p, q, r aren't all even, is zero. Consider for example $\int_{\Sigma} x^3 y^3 z^2 \, dA$: intuitively, the integral looks exactly the same on the $x \geq 0$ and $x \leq 0$ hemispheres, except that on one side the integrand

is exactly the negative of what it is on the other. Therefore the two contributions cancel and $I = 0$. If we were only integrating over the $x \geq 0$ hemisphere then this trick wouldn't work with x , but it still would with y . However if we integrated over the quarter-sphere given by $x \geq 0, y \geq 0$, then the integrand is positive everywhere, so the integral I will not vanish.

(The precise requirement for a trick like this to work is that there is some kind of spatial symmetry, such as the reflection in the yz -plane given by $(x, y, z) \mapsto (-x, y, z)$, which carries the domain of integration Σ to itself but under which the integrand changes sign. If you apply the change of variables formula (writing for example $x' = -x, y' = y, z' = z$) you'll get a new integral I' which computes the same number as I . But you'll see that, manifestly, $I' = -I$; and hence $I = 0$.)

3. In lectures I occasionally used a nice trick for calculating an integrals of a quadratic function over a sphere Σ . By rotational symmetry we have $\int_{\Sigma} x^2 \, dA = \int_{\Sigma} y^2 \, dA = \int_{\Sigma} z^2 \, dA$ and so each of these is $\frac{1}{3}$ of the value of $\int_{\Sigma} (x^2 + y^2 + z^2) \, dA$ which is just the radius-squared times the area of the sphere.

4. Change of variables, and/or reparametrisation. Say you are doing an integral over a surface Σ which is the graph of a function $z = g(x, y)$ whose domain is a *disc* in the xy -plane. Once you get down to an integral of the form $\iint_D f(x, y) \, dx dy$, you might well want to change variables to polar coordinates in order to evaluate this.

(It will amount to precisely the same thing if you actually *start* by writing down a parametrisation of Σ using polar coordinates of the form $(r, \theta) \mapsto (r \cos \theta, r \sin \theta, g(r \cos \theta, r \sin \theta))$ and carrying out the computation using $\mathbf{T}_r \times \mathbf{T}_\theta$, etc. Most people probably find this second way of thinking more complicated, however.)

5. A vector integral can always be treated as a scalar integral using the formula

$$\int_{\Sigma} \mathbf{v} \cdot d\mathbf{A} = \int_{\Sigma} (\mathbf{v} \cdot \hat{\mathbf{n}}) \, dA.$$

This is only useful when the surface Σ is geometrically so simple (e.g. when Σ is part of a sphere, cylinder, or plane) that you can *use geometry to write down* a formula for its unit normal $\hat{\mathbf{n}}(x, y, z)$. When we do this, if the integrand $\mathbf{v} \cdot \hat{\mathbf{n}}$ turns out to be a very simple function (for example a constant or a function we can handle using a symmetry trick), we might be able to evaluate the integral without even parametrising the surface!

Note. If you'd need to *calculate* $\hat{\mathbf{n}}$ by parametrising the surface and using the formula

$$\hat{\mathbf{n}} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|},$$

then there's *no point* in trying to use this trick, because you will only end up saying

$$\int_{\Sigma} \mathbf{v} \cdot d\mathbf{A} = \int_{\Sigma} (\mathbf{v} \cdot \hat{\mathbf{n}}) \, dA = \iint_D \left(\mathbf{v} \cdot \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} \right) \|\mathbf{T}_u \times \mathbf{T}_v\| \, du \, dv = \iint_D \mathbf{v} \cdot (\mathbf{T}_u \times \mathbf{T}_v) \, du \, dv$$

which just brings you, full circle, back to the standard formula for the vector integral! This little calculation is actually the *proof* that the vector and scalar integrals do in fact compute the same thing, but it's of no practical use in calculation.

In any case, note that when you compute a surface integral using a parametrisation, the fact that the formula for the scalar integral involves the *magnitude* $\|\mathbf{T}_u \times \mathbf{T}_v\|$, with its unpleasant *square root*, means that scalar integrals actually tend to be more difficult than vector ones!

D. Using Stokes' theorem and the other "fundamental theorems of calculus"

These are tricks which involve changing the *dimension* of the thing we're integrating over. Stokes' theorem can be used in either direction to replace one integral by the other:

$$\int_{\Sigma} (\nabla \times \mathbf{v}) \cdot d\mathbf{A} \longleftrightarrow \int_{\partial\Sigma} \mathbf{v} \cdot d\mathbf{s}$$

but you should be careful not to make it say something it doesn't!

1. Make sure that you only try to use it in the "left to right" direction *when the integrand is the curl of something*: Stokes' Theorem *DOES NOT SAY* that $\int_{\Sigma} \mathbf{v} \cdot d\mathbf{A} = \int_{\partial\Sigma} \mathbf{v} \cdot d\mathbf{s}$! This direction is typically useful because it reduces the dimension of the integral, which often makes evaluation easier.

2. A variation of this left-to-right direction is to *change the surface* Σ to a simpler surface Σ' with the same boundary as Σ , and then compute $\int_{\Sigma'} (\nabla \times \mathbf{v}) \cdot d\mathbf{A}$ instead of $\int_{\Sigma} (\nabla \times \mathbf{v}) \cdot d\mathbf{A}$; the theorem tells us they are both equal to $\int_{\partial\Sigma} \mathbf{v} \cdot d\mathbf{s}$ and so equal to one another. This trick might be useful if $\partial\Sigma$ is a complicated curve, but there is a simple choice of surface Σ' .

3. The "right to left" direction can only be used if you begin with an integral of the form $\int_{\gamma} \mathbf{v} \cdot d\mathbf{s}$ where γ is a *closed* oriented curve, because you have to be able to *choose* an oriented surface Σ whose boundary agrees with γ . Because this direction increases the dimension of the integral, it tends to be useful only if $\nabla \times \mathbf{v}$ is a lot simpler than \mathbf{v} , or if a very nice surface Σ can be chosen (say, one which is flat).

4. The same sort of comments apply to using Gauss' Theorem. You *need the divergence in the integrand* to go left-to-right, and you have to start with a *closed* surface to go from right to left.

$$\int_R (\nabla \cdot \mathbf{v}) dV \longleftrightarrow \int_{\partial R} \mathbf{v} \cdot d\mathbf{A}$$

5. Similarly for the gradient version: you *need the gradient in the integrand* to go left-to-right, but can always compute a potential difference between two points by integrating its gradient along a curve between them. (Here γ_+ is the endpoint and γ_- the starting point of the oriented curve γ .)

$$\int_{\gamma} \nabla\phi \cdot d\mathbf{s} \longleftrightarrow \phi(\gamma_+) - \phi(\gamma_-)$$

6. Green's theorem is simply the 2d version of Stokes' theorem: for a vector field $\mathbf{v}(x, y) = (P(x, y), Q(x, y))$ and D a region in the plane,

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{\partial D} \mathbf{v} \cdot d\mathbf{s}$$

A useful trick is that choosing $Q = x, P = 0$ (or $Q = 0, P = -y$, or $Q = \frac{1}{2}x, P = -\frac{1}{2}y$, or ...) makes the scalar curl $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ so that $\int_{\partial D} \mathbf{v} \cdot d\mathbf{s}$ then computes the *area* of D .