SOLUTION TO SELECTED HOMEWORK PROBLEM

Several students asked me about Problem 5 in Section 7.6, page 497. So here is an outline of the solution. Although by now, we know an easier how to do it (see below), it is still instructive knowing how to do it the direct way. In any case, if you had to do the integral of **F** instead of the curl $\nabla \times \mathbf{F}$, you probably would have to parametrize it anyway.

Problem Evaluate $\int\int_S (\nabla \times F) \cdot d\mathbf{S}$, where S is the surface $x^2 + y^2 + 3z^2 = 1$, $z \leq 0$, and $\mathbf{F}$ is the vector field $\mathbf{F}(x, y, z) = (y, -x, zx^3y^2)$, with the normal vector $\mathbf{n}$ pointing upwards.

Solution 1 : (Direct method) We calculate

$$(\nabla \times \mathbf{F})(x, y, z) = (2yzx^3, -3zx^2y^2, -2).$$

To parametrize S, observe that we can view S as part of the graph of the function $z = -\sqrt{(1 - x^2 - y^2)/3}$ (just solve for the z coordinate, keeping in mind that $z \leq 0$). Hence we get the parametrization

$$T(x, y) = (x, y, -\sqrt{(1 - x^2 - y^2)/3}), \quad x^2 + y^2 \leq 1.$$

(You can visualize S as a bowl below the xy plane, whose intersection with the xy plane is the unit circle). We get from this the tangent vectors

$$T_x = (1, 0, \frac{-x}{-3\sqrt{(1 - x^2 - y^2)/3}}), \quad T_y = (0, 1, \frac{y}{-3\sqrt{(1 - x^2 - y^2)/3}}).$$

Hence we get

$$T_x \times T_y = (\frac{-x}{\sqrt{3(1 - x^2 - y^2)}}, \frac{-y}{\sqrt{3(1 - x^2 - y^2)}}, 1).$$

The vector $T_x \times T_y$ points upwards, as its z-coordinate is positive. Hence we have the orientation as required. Let D denote the unit disk $x^2 + y^2 \leq 1$ in $\mathbf{R}^2$. Then we can set up the integral as

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \;=\; \int\int_D (\nabla \times \mathbf{F})(x, y, , -\sqrt{(1 - x^2 - y^2)/3}) \cdot (T_x \times T_y)\ dx\ dy$$

Using the formula for $(\nabla \times \mathbf{F})(x, y, z)$ at the beginning, we obtain for the integral

$$\int\int_D (-2x^3y\sqrt{(1 - x^2 - y^2)/3}, 3x^2y^2\sqrt{(1 - x^2 - y^2)/3}, -2) \cdot (\frac{-x}{\sqrt{3(1 - x^2 - y^2)}}, \frac{-y}{\sqrt{3(1 - x^2 - y^2)}}, 1)\ dx\ dy$$

Evaluating the dot product, we obtain

$$\int\int_D 2x^4y(1 - x^2 - y^2)/3 - x^2y^3(1 - x^2 - y^2) - 2\ dx\ dy.$$

Integrating over the disk, we use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and $dx\ dy = r\ dr\ d\theta$. Then our integral becomes

$$\int_0^{2\pi}\int_0^1 (\frac{2}{3}r^5\cos^4\theta\sin\theta(1 - r^2) - r^5\cos^2\theta\sin^3\theta(1 - r^2) - 2)r\ dr\ d\theta.$$

Observe that we are allowed to integrate with respect to θ first. Moreover, we have

$$\int_0^{2\pi} \cos^4\theta\sin\theta\ d\theta \;=\; -\frac{1}{5}\cos^5\theta|_0^{2\pi} \;=\; 0,$$

$$\int_0^{2\pi} \cos^2\theta \sin^3\theta \; d\theta \;=\; \int_0^{2\pi} \cos^2\theta(1-\cos^2\theta)\sin\theta \; d\theta \;=\; \int_0^{2\pi} \cos^2\theta \sin\theta \; d\theta \;-\; \int_0^{2\pi} \cos^4\theta \sin\theta \; d\theta,$$

and one shows as in the first case that this integral has to be equal to 0. Hence we can ignore the first two complicated terms of the integral and obtain as result the integral

$$\int_0^{2\pi}\int_0^1 -2r \; dr \; d\theta \;=\; -2\pi.$$

Solution 2 : The problem becomes significantly easier if you use Stokes' theorem. Observe that the boundary of S is the unit circle C in the xy plane. Draw a picture for yourself (using the characterization of S as a bowl, see Solution 1) to convince yourself that the correct orientation for the circle, as required for Stokes' theorem is counterclockwise. Hence we can paramtrize C as usual by $c(t) = (\cos t, \sin t, 0)$, and we obtain

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \;=\; \int_C \mathbf{F} \cdot ds \;=\; \int_0^{2\pi} \mathbf{F}(\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0) \; dt \;=$$

$$=\; \int_0^{2\pi} (\sin t, -\cos t, 0) \cdot (-\sin t, \cos t, 0) \; dt \;=\; \int_0^{2\pi} -\sin^2 t - \cos^2 t \; dt \;=-2\pi.$$