No h'm of x-simple / y-simple region

Def: region \( D \subset \mathbb{R}^2 \) called y-simple

"if region is between graphs of functions \( \phi_1(x) \) and \( \phi_2(x) \)"

Precise: If \( \exists \alpha < b \) numbers and functions \( \phi_1, \phi_2 : [\alpha, b] \rightarrow \mathbb{R} \) such that
\[
D = \{(x, y) : \alpha \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\}.
\]
Many regions are both $x$-simple and $y$-simple.

**Examples**

(a) $O$ - triangle with corners $(0,0)$, $(0,1)$, $(1,0)$

\[
\begin{align*}
0 & \leq x \leq 1 \\
0 & \leq y \leq 1-x
\end{align*}
\]

Equation of line: $x+y=1$ \Rightarrow $y=1-x$
0. upper semidish of radius 1

\[ x^2 + y^2 = 1 \]

\[ \Rightarrow y = \sqrt{1-x^2} \]

**y-simple:** possible x-values:

\(-1 \leq x \leq 1\)

0 \leq y \leq \sqrt{1-x^2}

**x-simple:** possible y-values:

\(-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}\)
Example: Calculate the integral
\[
\int_{0}^{a} \int_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2-y^2} \, dy \, dx
\]
\[
\int_{0}^{\sqrt{a^2}} \sqrt{a^2-y^2} \, dy = \frac{\pi}{2} a
\]

key observation: region is both x-simple and y-simple.
Can also express \( D \) as

\[
0 \leq y \leq a \\
0 \leq x \leq \sqrt{a^2 - y^2}
\]

\( \Rightarrow \) integral =

\[
\iiint_0^a \sqrt{a^2 - y^2} \, dx \, dy
\]

\[
= \int_0^a \left[ \sqrt{a^2 - y^2} \right]_{x=0}^{x=\sqrt{a^2 - y^2}} \, dy
\]

\[
= \int_0^a \sqrt{a^2 - y^2} \cdot \sqrt{a^2 - y^2} \, dy = \int_0^a (a^2 - y^2) \, dy
\]

\[
dx \cdot dy \text{ in different order as in previous page}
\]
Mean Value Inequality:

1-dim case: assume we have numbers \( m, M \)
such that \( m \leq f(x) \leq M \) for \( a \leq x \leq b \)

\[
\Rightarrow \quad m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a)
\]

2-dim case

\( f: D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \), assume \( m \leq f(x,y) \leq M \)
for all \((x,y)\) in \(D\)

\[
\Rightarrow \quad m \text{ area}(D) \leq \iint_D f(x,y) \, dA \leq M \text{ area}(D)
\]
Example: Let \( f(x,y) = \frac{1}{\sqrt{1 + 2x^8 + y^{12}}} \)

Let \( D = [0,1] \times [0,1] \)

Use mean value inequality to estimate \( \iint_D f(x,y) \, dA \)

Sol.

Smallest possible value of \( f \) in \( D \)

For largest possible value in denominator:

\[ 1 + 2x^8 + y^{12} \leq 1 + 2 \cdot 1^8 + 1^{12} = 4 \]

\[ 1 \leq \frac{1}{1} \quad \text{and} \quad 1 \leq 1 \]

\[ \Rightarrow \quad m = \frac{1}{\sqrt{4}} = \frac{1}{2} \quad \text{smallest value} \]

Largest value \( \Leftrightarrow \) smallest value of denominator at \( x=0 \) \( y=0 \)

\[ M = \frac{1}{\sqrt{1}} = 1 \]