Green's Theorem: relates a double integral over region \( D \subset \mathbb{R}^2 \) to a line integral over its boundary \( C \).

Stokes' Theorem is a generalization of Green's Theorem to surfaces \( S \subset \mathbb{R}^3 \) with boundary curve \( C \).
Recall: If $F : \mathbb{R}^3 \to \mathbb{R}^3$ vector field

$$\Rightarrow \text{curl } F = \nabla \times F$$

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

**Stokes' Theorem for graphs**

Let $S$ be the graph of function $f : D \to \mathbb{R}^3$ given by points $(x, y, z)$ where $z = f(x, y)$ and $(x, y)$ in $D$

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field

$$\Rightarrow \int_S \text{curl } F \cdot dS = \int_C F \cdot ds$$

integrals of vector fields depend on orientation!

Theorem only valid for compatible orientation.
• for surface integrals we need to specify which side is the positive side.
  Here: positive side = upper side.

• for line integrals: need to specify direction in which we run through curve.
  Here: need to walk on positive side of S on C such that S is to our left.

Example: Use Stokes' Theorem to calculate

$$\int_C \mathbf{F} \cdot d\mathbf{s}$$

where

C is the intersection of $2x + 2y + z = 2$

writen cylinder $x^2 + y^2 = 1$

oriented counterclockwise around 2-axis.
Solution: we apply Stokes' Theorem to the part of the plane $2x + 2y + z = 2$ which is inside the cylinder.

Parametrize $S$: Solve for $z$:

$z = 2 - 2x - 2y$
$x^2 + y^2 \leq 1$

Recall: normal vector for graph given by $T_x \times T_y = \left( -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \right)$

Here: $g(x,y) = 2 - 2x - 2y \Rightarrow \frac{\partial g}{\partial x} = -2, \frac{\partial g}{\partial y} = -2 \Rightarrow T_x \times T_y = (2, 2, 1)$
have checked: orientations are compatible  
(walking around counter clockwise 
compatible with normal vector pointing upwards) 
(i.e. its z-coordinate is positive)  

Stokes Theorem \Rightarrow 

\[ \int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S} \]

\( S \) parameterized by \( \Phi(u,v) = (u,v, 2-2u-2v) \)

\[ = \iint_{u^2+v^2<1} \text{curl} \mathbf{F} \cdot (u,v, 2-2u-2v) \cdot (2,2,1) \, du \, dv = \]

\( F(x,y,z) = (-y^3, x^3, -z^3) \)

\[ \text{Curl} \mathbf{F}: \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{array} \right| = \nabla (0) - \nabla (3x^2 + 3y^2) \]
\[
\iint_{u^2 + v^2 \leq 1} \left( 0, 0, 3u^2 + 3v^2 \right) \cdot (2, 2, 1) \, du \, dv
\]
\[
= \iiint_{u^2 + v^2 \leq 1} 3u^2 + 3v^2 \, du \, dv
\]
\[
= \frac{3\pi}{2}
\]

Polar coordinates:

\[u = r \cos \theta\]
\[v = r \sin \theta\]
Stokes' Theorem For General Case:

\[
S \subset \mathbb{R}^3 \text{ surface parametrized by} \quad \Phi \colon \Omega \to \mathbb{R}^3
\]

\( S \) has boundary curve \( C \)

\( F : \mathbb{R}^3 \to \mathbb{R}^3 \) a vector field

\[\int_C F \cdot ds = \iint_S \text{curl } F \cdot dS \uparrow \text{ normal vector}\]

if orientations are compatible

i.e. if we walk on \( C \) on the side of normal vector \( T_u \times T_v \)

\( \Rightarrow \) need to walk in direction s.t. \( S \) is to our left

\( \Rightarrow \) determines orientation for \( C \)
e.g. $T_u 	imes T_v$