Math 130 A HW 8 Solutions

6.2.2) $\dot{x} = y \qquad \dot{y} = -x + (1 - x^2 - y^2)y$

a) It's clear that both $\dot{x}$ and $\dot{y}$ are continuous on the open disk $x^2 + y^2 < 4$ as they're polynomials and so are the partial derivatives of $\dot{x}$ and $\dot{y}$ are continuous as well since they're polynomials. So the uniqueness and existence theorem is satisfied.

[1 point]

b) Using $x(t) = \sin t$ and $y(t) = \cos t$

must show → $\dot{x}(t) = \cos t$ and $\dot{y}(t) = -\sin t$

$\dot{x}(t) = y(t) = \cos t$ and $\dot{y}(t) = -x + (1 - x^2 - y^2)y$

$\Rightarrow \dot{y}(t) = -\sin t + (1 - \sin^2 t - \cos^2 t)\cos t$

$= -\sin t + (1 - \underbrace{(\cos^2 t + \sin^2 t)}_{1})\cos t = -\sin t$

[1 point]

c) From b) since $x(t) = \sin t$ and $y(t) = \cos t$ are solutions we know this forms the equation for a closed orbit ($x(t)^2 + y(t)^2 = 1 = \text{const.}$) which encloses the point $(\frac{1}{2}, 0)$. From the uniqueness theorem trajectories can't intersect and so if $x(0) = \frac{1}{2}$ and $y(0) = 0$ it follows the trajectory must satisfy $x^2 + y^2 < 1$. [1 point]

6.3.2) $\dot{x} = \sin y \qquad \dot{y} = x - x^3$

Find fixed points $\quad \dot{x} = 0 = \sin(y) \Rightarrow y = n\pi$ where $n \in \mathbb{Z}$

$\dot{y} = 0 = x - x^3 \Rightarrow x = 0, \pm 1$ [1 point for fixed points]

linearize around the fixed points using the Jacobian

$$A_{(x,y)} = \begin{bmatrix} 0 & \cos y \\ 1-3x^2 & 0 \end{bmatrix} \qquad A_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_{(\pm 1,0)} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \qquad A_{(0,\pi)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad A_{(\pm 1,\pi)} = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}$$

(except $y=0$, $y=\pi$ are sufficient as further y values are repetitive)

Classify fixed points

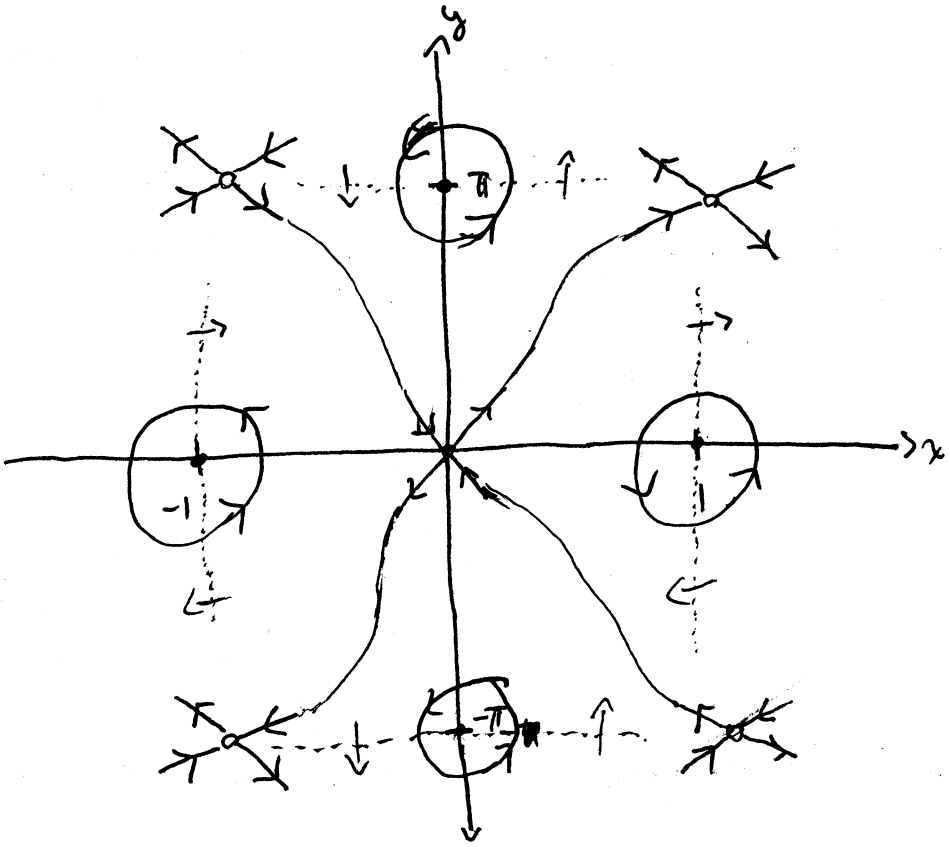
(x,y)			
For $(0,0)$	$\lambda = \pm 1 \Rightarrow$ Saddle point	ξ-vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	[1 point for each classification]
$(\pm 1, 0)$	$\lambda = \pm\sqrt{2}i \Rightarrow$ center		
$(0, \pi)$	$\lambda = \pm i \Rightarrow$ center		
$(\pm 1, \pi)$	$\lambda = \pm\sqrt{2} \Rightarrow$ saddle point	$\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$	
$(\pm 1, -\pi)$	$\lambda = \pm\sqrt{2} \Rightarrow$ Saddle	$\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$	

<u>Nullclines</u>

horizontal - $\dot{y} = 0 \Rightarrow x = 0, \pm 1$

vertical - $\dot{x} = 0 \Rightarrow y = n\pi$

Using all of the information we draw the phase portrait



[1 point for drawing all fixed points and 1 point for approximate trajectories]

Total: 7 points

6.5.1) $\ddot{x} = x^3 - x$

a) $\dot{x} = y \qquad \dot{y} = x^3 - x$

$\dot{x} = 0 \Rightarrow y = 0 \qquad \dot{y} = 0 = x^3 - x \Rightarrow x = 0, \pm 1$

Linearize around fixed points and find eigenvalues and eigenvectors

$$A_{(x,y)} = \begin{bmatrix} 0 & 1 \\ 3x^2 - 1 & 0 \end{bmatrix} \qquad A_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad A_{\pm 1,0} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$\lambda = \pm i$
center

$\lambda = \pm \sqrt{2}$
Saddle point
$\left(\frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}\right)$

[1 point for each classification]

b) One approach to this is to consider the system as a force and potential system with $F(x) = \ddot{x} = -\frac{dV}{dx}$

$$\Rightarrow V(x) = \int x^3 - x \, dx = \frac{x^2}{2} - \frac{x^4}{4}$$

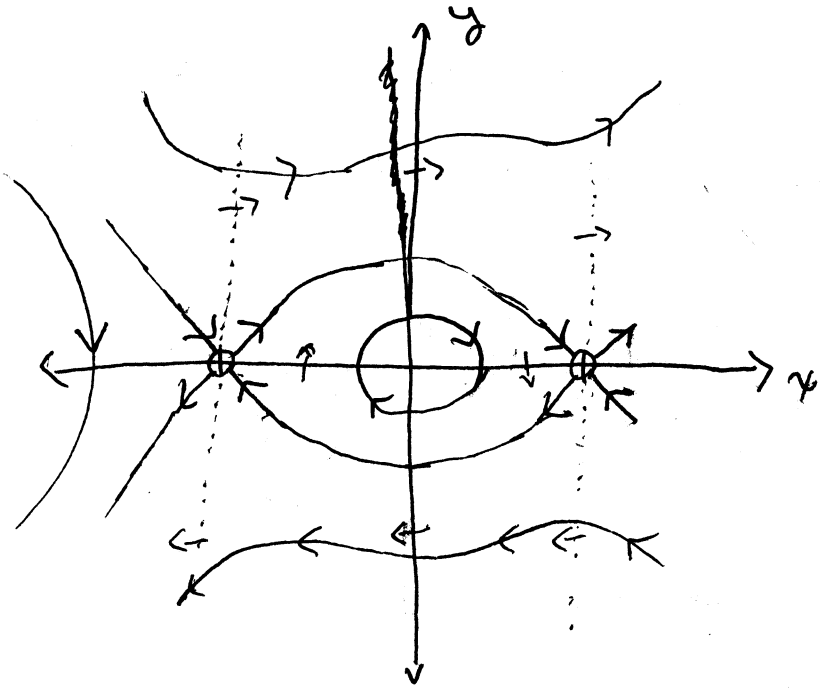
Using the equation for total energy $E = \frac{1}{2} m \dot{x}^2 + V(x)$

$$\Rightarrow E = \frac{1}{2} y^2 + \frac{1}{2} x^2 - \frac{1}{4} x^4$$ is a conserved quantity (can check by taking time derivative)

[1 point for correct quantity]

nullclines: vert – $y=0$
horiz – $x=0$, $x=\pm 1$

c)



[1 point for drawing all fixed points and 1 point for approximate trajectories]

Total: 5 points