Section 5.1, Problem 9

For this problem, we use the Neumann BC formula for solutions to the wave equation, which is similar to the Dirichlet BC formula derived in the previous HW. However, there is a very important difference that has to do with the presence of a constant term in the expansion. This results in the separation of variables ODE:

$$T_0'' \ = \ 0 \ , \tag{1}$$

for the time component of the solution $u_0 = T_0 X_0$, were $X_0 = 1$ (this clearly is a solution to $-X_0'' = \lambda_0 X_0$ with $\lambda_0 = 0$). The general solution to (1) is:

$$T_0(t) \ = \ a + bt \ .$$

Thus, for a full solution to the Neumann wave equation on $[0, \ell]$ we have:

$$u(x,t) \ = \ A_0 + B_0 t + \sum_{1\leqslant n} \Big(A_n \cos(\frac{cn\pi}{\ell} t) + \frac{\ell B_n}{cn\pi} \sin(\frac{cn\pi}{\ell} t) \Big) \cos(\frac{n\pi}{\ell} x) \ ,$$

Where the initial data is given by:

$$u(x,0) \ = \ A_0 + \sum_{1\leqslant n} A_n \cos(\frac{n\pi}{\ell} x) \ ,$$
$$u_t(x,0) \ = \ B_0 + \sum_{1\leqslant n} B_n \cos(\frac{n\pi}{\ell} x) \ .$$

In our particular problem, we can use the expansion of the function $\cos^2(x)$ on the interval $[0, \pi]$ from Problem # 2 above to write:

$$u(x,t) \ = \ \frac{1}{2} t + \frac{1}{4c} \sin(2ct) \cos(2x) \ .$$

Section 5.2, Problem # 2

Recall that a function $\phi(x)$ is periodic of period ℓ if $\phi(x+\ell) = \phi(x)$. We could also ask if ℓ is the *least* period, meaning that ℓ is as small as possible. If $\phi(x) = \cos(x) + \cos(\alpha x)$ with $\alpha = \frac{p}{q}$ rational (here $\gcd(p,q) = 1$), then if $\ell = 2\pi q$ we clearly have:

$$\phi(x+\ell) \ = \ \cos(x + 2\pi q) + \cos(\alpha x + 2\pi p) \ = \ \cos(x) + \cos(\alpha x) \ = \ \phi(x) \ .$$

Its also very easy to see that this choice of ℓ is optimal. Notice that by periodicity any period ℓ must be such that:

$$\phi(\ell) \ = \ \phi(0) \ = \ 2 \ .$$

Since $|\cos(x)| \leqslant 1$, this means that we must have both:

$$\cos(\ell) \ = \ 1 \ , \qquad \qquad \cos(\alpha \ell) \ = \ 1 \ .$$

therefore, we must have $\ell = 2\pi k_1$ and $\frac{p}{q}\ell = 2\pi k_2$ for some $k_1, k_2 \in \mathbb{N}$. This gives the relation:

$$p k_1 \ = \ q k_2 \ .$$

Since $\gcd(p,q) = 1$ by design, using a little elementary arithmetic we see that we must have $k_1 = q$ and $k_2 = p$, which is the same thing as $\ell = 2\pi q$.

Section 4.3, Problem # 8

This problem turned out to be trickier than intended. We give here an outline of the solution, using the results of the book.

(a) Solving for a_ℓ, we get $a_\ell = -a_0/(1+a_0\ell)$. This can be rewritten as $1 = (1+\ell a_0)(1+\ell a_\ell)$. Check that its graph is the usual hyperbola $xy = 1$ with its center shifted from $(0,0)$ to $(-1/\ell, -1/\ell)$. Draw the hyperbola!

(b) The hyperbola divides the $a_0 a_\ell$ plane into the following regions

(i) $a_0 < -1/\ell$ and $a_l < -a_0/(1+a_0 l)$ (or $a_0 + a_\ell < -a_0 a_\ell$)
(ii) $a_0 > -1/\ell$, $a_l > -a_0/(1+a_0 l)$ (or $a_0 + a_\ell < -a_0 a_\ell$),
(iii) the remainder between the two branches of the hyperbola.

Recall that the number of negative eigenvalues is equal to the number of intersections between the graphs of the functions

$$\tanh \gamma l \quad \text{and} \quad g(\gamma) = -\frac{(a_0 + a_\ell)\gamma}{\gamma^2 + a_0 a_\ell}.$$

They have the derivatives

$$\ell/\cosh^2 \gamma\ell \quad \text{and} \quad g'(\gamma) = -\frac{(a_0 + a_\ell)(a_0 a_\ell - \gamma)}{(\gamma^2 + a_0 a_\ell)^2}.$$

Their derivatives coincide at $\gamma = 0$ if and only if $a_0 + a_\ell = -a_0 a_\ell \ell$, in which case 0 is an eigenvalue.

In the case $a_0 > 0, a_\ell > 0$ we have no negative eigenvalue, since $\tanh \gamma l$ is positive, while $g(\gamma)$ is negative (for $\gamma > 0$). The case $a_0 < 0, a_\ell > 0$ is discussed in the book on page 95. By symmetry of the formulas for a_0 and a_ℓ, we get the same result also for the case $a_0 > 0, a_\ell < 0$.

It remains to consider the case $a_0 < 0, a_\ell < 0$. Here the function $g(\gamma)$ tends to 0 if $\gamma \to \infty$ and if $\gamma \to 0$. We see from the formula for the derivative that $g'(\gamma) = 0$ if and only if $\gamma = \sqrt{a_0 a_\ell}$ ($a_0 + a_\ell \neq 0$ in this case!). One calculates

$$g(\sqrt{a_0 a_\ell}) = -\frac{a_0 + a_\ell}{2\sqrt{a_0 a_\ell}} \geq 1,$$

where the last inequality comes from the fact that $\sqrt{ab} \leq (a+b)/2$ for any two positive numbers a and b, i.e. geometric mean $\leq$ arithmetic mean. Hence the maximum of g is above $\tanh \gamma\ell$, while $g(\gamma) < \tanh \gamma\ell$ if $\gamma \to \infty$ and if $\gamma \to 0$. This finishes all cases.

Section 5.2, Problem # 12

This problem can be solved by finding the series for e^x and e^{-x} and then taking their average. This computation is most easily done in complex form. The series will look like:

$$e^x = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{\ell}x} ,$$

where the coefficients c_n are given be the integrals:

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\ell}^{\ell} e^x e^{-i\frac{n\pi}{\ell}x}\, dx , \\
&= \frac{1}{2\pi}\int_{-\ell}^{\ell} e^{(-i\frac{n\pi}{\ell}+1)x}\, dx , \\
&= \frac{1}{2\pi(-i\frac{n\pi}{\ell}+1)}\left(e^{(-i\frac{n\pi}{\ell}+1)\ell} - e^{-(-i\frac{n\pi}{\ell}+1)\ell}\right) , \\
&= \frac{(-1)^n}{\pi(-i\frac{n\pi}{\ell}+1)}\sinh(\ell) , \\
&= (-1)^n\frac{\sinh(\ell)}{\frac{n^2\pi^2}{\ell^2}+1} + i(-1)^n\frac{\frac{n\pi}{\ell}\sinh(\ell)}{\frac{n^2\pi^2}{\ell^2}+1} .
\end{aligned}$$

Plugging this last expression into the complex series, and then using the DeMoivre formula to expand into trig functions, we have after some eliminations that:

$$e^x = \sum_{n=-\infty}^{\infty}(-1)^n\frac{\sinh(\ell)}{\frac{n^2\pi^2}{\ell^2}+1}\cos(\frac{n\pi}{\ell}x) - \sum_{n=-\infty}^{\infty}(-1)^n\frac{\frac{n\pi}{\ell}\sinh(\ell)}{\frac{n^2\pi^2}{\ell^2}+1}\sin(\frac{n\pi}{\ell}x) .$$

Taking the even part of this last expression, we have that:

$$\begin{aligned}
\cosh(x) &= \frac{e^x+e^{-x}}{2} , \\
&= \sum_{n=-\infty}^{\infty}(-1)^n\frac{\sinh(\ell)}{\frac{n^2\pi^2}{\ell^2}+1}\cos(\frac{n\pi}{\ell}x) , \\
&= \sinh(\ell) + 2\sum_{1\leqslant n}(-1)^n\frac{\sinh(\ell)}{\frac{n^2\pi^2}{\ell^2}+1}\cos(\frac{n\pi}{\ell}x) .
\end{aligned}$$

Section 5.3, Problem 2

We'll solve each part of this problem separately:

a) We are trying to show that $\langle 1, x \rangle = 0$. That is:

$$\int_{-1}^{1} 1 \cdot x \; dx \;\; = \;\; 0 \; .$$

This is a simple computation.

b) Next, we are trying to find $p_2(x) = a + bx + cx^2$ such that $\langle 1, p_2(x) \rangle = \langle x, p_2(x) \rangle = 0$. That is, we need both:

$$\int_{-1}^{1} (a + bx + cx^2) \; dx \;\; = \;\; 0 \; ,$$

$$\int_{-1}^{1} x \cdot (a + bx + cx^2) \; dx \;\; = \;\; 0 \; ,$$

which gives the system:

$$2a + \frac{2}{3}c \;\; = \;\; 0 \; ,$$

$$\frac{2}{3}b \;\; = \;\; 0 \; .$$

One solution is $a = 1, c = -3$, and $b = 0$. this gives:

$$p_2(x) \;\; = \;\; 1 - 3x^2 \; .$$

c) Finally, we are trying to find $p_3(x) = a + bx + cx^2 + dx^3$ with the property that:

$$\int_{-1}^{1} (a + bx + cx^2 + dx^3) \; dx \;\; = \;\; 0 \; ,$$

$$\int_{-1}^{1} x \cdot (a + bx + cx^2 + dx^3) \; dx \;\; = \;\; 0 \; ,$$

$$\int_{-1}^{1} x^2 \cdot (a + bx + cx^2 + dx^3) \; dx \;\; = \;\; 0 \; .$$

This yields the system:

$$2a + \frac{2}{3}c \;\; = \;\; 0 \; ,$$

$$\frac{2}{3}b + \frac{2}{5}d \;\; = \;\; 0 \; ,$$

$$\frac{2}{3}a + \frac{2}{5}c \;\; = \;\; 0 \; .$$

In this case, a specific solution is $a = c = 0$ and $b = 3, d = -5$. This gives:

$$p_3(x) \;\; = \;\; 3x - 5x^3 \; .$$

SECTION 5.3, PROBLEM 8.

We suppose here that both X_1 and X_2 obey the Robin condition:

$$\begin{aligned} X'(0) &= a_0 X(0) \ , \\ X'(\ell) &= -a_\ell X(\ell) \ . \end{aligned}$$

Therefore, we have the string of identities (we'll set $[a,b] = [0,\ell]$ here):

$$\begin{aligned} \left(-X_1'X_2 + X_1X_2'\right)\Big|_0^\ell &= -X_1'(\ell)X_2(\ell) + X_1(\ell)X_2'(\ell) + X_1'(0)X_2(0) - X_1(0)X_2'(0) \ , \\ &= a_\ell X_1(\ell)X_2(\ell) + X_1(\ell)\left(-a_\ell X_2(\ell)\right) + a_0 X_1'(0)X_2(0) - X_1(0)a_0X_2'(0) \ , \\ &= 0 \ . \end{aligned}$$

SECTION 5.3, PROBLEM 12.

This is just basic integration by parts. That is, we simply integrate both sides identity:

$$f''(x)g(x) = [f'(x)g(x)]' - f'(x)g'(x) \ ,$$

and use the fundamental theorem of calculus for the first term on the right hand side (that is $\int_a^b F' = F|_a^b$).

SECTION 5.3, PROBLEM 13.

This is very direct. Suppose that $X(x)$ is an eigenfunction such that:

$$X(x)X'(x)\Big|_{x=a}^{x=b} \leqslant 0 \ . \tag{14}$$

Then we have that:

$$\begin{aligned} \lambda\int_a^b X(x)X(x)\ dx &= -\int_a^b X''(x)X(x)\ dx \ , \\ & -X(x)X'(x)\Big|_{x=a}^{x=b} + \int_a^b X'(x)X'(x)\ dx \ . \end{aligned}$$

If (14) is true, then this last line *must* be strictly positive unless $X' = 0$. The case of $X' = 0$ can safely be ruled out because it implies that $\lambda = 0$ (we only care about dealing with cases where we may have $\lambda < 0$ here). Thus, we have that:

$$\lambda\int_a^b X(x)X(x)\ dx = \{\text{Something strictly positive}\} \ .$$

But we may also assume that $X \neq 0$ for all x. Thus we may divide through by the strictly positive quantity $\int_a^b X(x)X(x)dx$ to yield the identity:

$$\lambda = \{\text{Something strictly positive}\}\ .$$

Section 5.4, problem 12.

We have calculated the sine series of $f(x) = x$ on the interval $[0, l]$ to be equal to (see Example 3 on p. 105)

$$\frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}.$$

On the other hand, we have $\int_0^l x^2\ dx = l^3/3$. By Parseval's equality, we get

$$l^3/3 = \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^l \sin^2\frac{n\pi x}{l}\ dx.$$

Using $\int_0^l \sin^2\frac{n\pi x}{l}\ dx = l/2$ and canceling l^3, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Section 6.1, problem 6.

In polar coordinates, we have $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$. The boundary condition is $u(a,\theta) = 0 = u(b,\theta)$ for $0 \le \theta \le 2\pi$. As the boundary conditions do not depend on θ, we assume u to be a function of r only. You can also check this independently by considering separable solutions $R(r)\Theta(\theta)$, where the given annulus corresponds to the rectangle $a \le r \le b$ and $0 \le \theta \le 2\pi$. Now the second order differential equation

$$R'' + \frac{1}{r}R' = 1$$

has the special solution $R(r) = r^2/4$. As the homogeneous ODE $R'' + \frac{1}{r}R' = 0$ has the general solution $R(r) = C_1\log(r) + C_2$, we obtain the general solution of the inhomogeneous ODE

$$R(r) = \frac{r^2}{4} + C_1\log(r) + C_2.$$

Checking the boundary conditions, we obtain for the constants

$$C_1 = \frac{b^2 - a^2}{4\log b/a} \quad \text{and} \quad C_2 = -\frac{(b^2-a^2)(\log a)}{4\log b/a} - \frac{a^2}{4}.$$

Section 6.1, problem 9.

a) After making a reduction to the radial variable (again the boundary conditions are symmetric) this question asks to find the solution u to the ODE problem:

$$\begin{aligned} \partial_r^2 u + \frac{2}{r}\partial_r u &= 0 \ , \\ u(1) &= 100 \ , \\ \partial_r u(2) &= -\gamma \ . \end{aligned}$$

The general solution to this radial ODE was derived in class and is $u(r) = \frac{C_1}{r} + C_2$. The second boundary condition gives us that $C_1 = 4\gamma$. Thus $u(r) = \frac{4\gamma}{r} + C$ for some C.

The first (inner) boundary condition then gives us that $C = 100 - 4\gamma$. Therefore, the desired solution is:

$$u(r) = \frac{4\gamma(1-r)}{r} + 100 \ .$$

b) In this setup, the temperature goes up as r decreases and vice-versa. Thus, the hottest temperature is when $r = 1$ and is $100^o C$. The coldest temperature is when $r = 2$ and is $(100 - 2\gamma)^o C$.

Notice that this makes sense because there is a net flux *out* of the shell at the outer boundary ($\partial_r u = -\gamma < 0$ there). Thus there must be a net heat flux *into* the inner boundary, and therefore the temperature it hotter there.

c) Yes, all you have to do is to choose $\gamma = 40$ in the equation $(100 - 2\gamma)$ for the temperature on the outer boundary.

Section 6.2, problem 3.

For this problem, we look for a solution which is obtained through separation of variables. That is, we look for a solution of the form:

$$u(x,y) = \sum_n X_n(x) Y_n(y) \ . \tag{18}$$

Since the only non-zero boundary condition here is in the y variable, we will choose:

$$Y_0 = \frac{1}{2} \ , \qquad\qquad Y_2 = \frac{1}{2}\cos(2y) \ .$$

Therefore, we need to simply choose our X_n such that they solve the following:

$$\begin{aligned} X_n'' &= n^2 X_n \ , \\ X_n(0) &= 0 \ , \\ X_n(\pi) &= 1 \ . \end{aligned}$$

If we do this, then we clearly get the correct solution by sticking things together according to formula (18). Solving the X_n system for $n = 0$ we have $X_0(x) = \frac{x}{\pi}$. This is easily seen because $X_0'' = 0$ means that X_0 is a linear function $X_0 = a + bx$, and a, b are determined by the endpoint conditions to be $a = 0$ and $b = \frac{1}{\pi}$.

Likewise, we solve for $X_2(x)$ by in general writing:

$$X_2(x) = a\cosh(2x) + b\sinh(2x) \ .$$

Choosing a, b so that $X_2(0) = 0$ and $X_2(\pi) = 1$ we have that $a = 0$ and $b = \frac{1}{\sinh(2\pi)}$. Thus, we have that $X_2(x) = \frac{\sinh(2x)}{\sinh(2\pi)}$.

Adding all of the pieces together, we have that:

$$u(x, y) = \frac{x}{2\pi} + \frac{\sinh(2x)}{2\sinh(2\pi)}\cos(2y) \ .$$