

Here are more homework solutions. If you want to know more about other homeworks, or you would like to see more details or have other questions, ask in class, review section or office hour.

#### SECTION 3.1, PROBLEM 4

For (a) and (b), we use equation (6) in Section 3.1 which says that the function  $v(x, t)$  defined by this formula is the unique (by results in Section 2) solution of the diffusion equation with the initial condition  $v(x, 0) = \phi(x)$ . Hence, in (a),  $v(x, t)$  is the solution of the diffusion equation  $v_t = kv_{xx}$  with the initial condition  $v(x, 0) = f(x)$ . Moreover, we showed in Section 2.4 that any linear combination of solutions of the diffusion equation and of their derivatives is again a solution of the diffusion equation. Hence  $w(x, t)$  is the solution of the diffusion equation with initial condition  $w(x, 0) = v_x(x, 0) - 2v(x, 0) = f'(x) - 2f(x)$ .

For part (c), just observe that  $g(x) = f'(x) - 2f(x) = -2x - 1$  for  $x < 0$ , and  $g(x) = f'(x) - 2f(x) = -2x + 1 = -g(-x)$  for  $x > 0$ . Part (d) now follows from this, part (b) and Exercise 24.11.

For part (e), it only remains to check that  $v(x, t)$  satisfies the Robin boundary condition. As  $w(x, t)$  is an odd function in  $x$  for  $t > 0$ , we have  $0 = w(x, 0) = v_x(0, t) - 2v(0, t)$ . This finishes the proof.

#### SECTION 3.2, PROBLEM 1

We are considering the problem

$$\begin{aligned} v_{tt} &= c^2 v_{xx} , \\ v(x, 0) &= \phi(x), \quad v_t(x, 0) = \psi(x) , \\ v_x(0, t) &= 0 . \end{aligned}$$

Here  $0 < x < \infty$  and  $-\infty < t < \infty$ . Essentially, we proceed as in Section 3.2 in the book, except for the changed boundary condition  $v_x(0, t) = 0$ . In analogy to our study of the diffusion equation, we reduce this problem to the one without boundary conditions by considering the *even* extension of this problem to the whole line. So we define  $\phi_{ev}(x) = \phi(|x|)$  and  $\psi_{ev}(x) = \psi(|x|)$ , for  $-\infty < x < \infty$ , and we consider the initial value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} , \\ u(x, 0) &= \phi_{ev}(x), \quad v_t(x, 0) = \psi_{ev}(x) , \end{aligned}$$

where now  $-\infty < x < \infty$ . Using the solution formula for the wave equation, we get

$$u(x, t) = \frac{1}{2}[\phi_{ev}(x + ct) + \phi_{ev}(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{ev}(y) dy.$$

We would like to express this solution for  $x > 0$  by just using the original functions  $\phi(x)$  and  $\psi(x)$ . The only nontrivial part is if the integration variable  $y$  could become negative, i.e. if

$x - ct < 0$ . Observe that in this case we have  $\int_{x-ct}^0 \psi_{ev}(y)dy = \int_0^{|x-ct|} \psi_{ev}(y)dy$ . Hence the general solution becomes

$$v(x, t) = \frac{1}{2}[\phi(x + ct) - \phi(|x - ct|)] + \begin{cases} \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y)dy & \text{if } x - ct \geq 0, \\ \frac{1}{2c} [\int_{|x-ct|}^{x+ct} \psi(y)dy + 2 \int_0^{|x-ct|} \psi(y)dy] & \text{if } x - ct < 0. \end{cases}$$

### SECTION 3.2, PROBLEM 2

Apply the solution of the previous problem, with  $\phi(x) = 0$  for all  $x \geq 0$  and with  $\psi(x) = 1$  for  $a < x < 2a$  and  $\psi(x) = 0$  otherwise. Ask me in class if you want to see the sketches.

### SECTION 3.3, PROBLEM 2

Following the hint, we consider the function  $V(x, t) = v(x, t) - h(t)$ , where  $v(x, t)$  is a solution of the given PDE. Then  $V(x, t)$  satisfies

$$\begin{aligned} V_t - kV_{xx} &= f(x, t) - h'(t), \\ V(0, t) &= 0, \\ V(x, 0) &= \phi(x) - h(0). \end{aligned}$$

This problem now has a homogeneous boundary condition, for which we can use our solution formula, see Section 3.1, (6). Hence we have

$$V(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}] (\phi(y) - h'(t)) dy.$$

Substituting  $p = (x - y)/\sqrt{4kt}$ , we get

$$\frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-(x-y)^2/4kt} dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4kt}} e^{-p^2} dp = \frac{1}{2}(1 + \mathcal{Erf}(x/\sqrt{4kt})).$$

Similarly substituting  $p = (x + y)/\sqrt{4kt}$ , we get

$$\frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-(x+y)^2/4kt} dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-x/\sqrt{4kt}} e^{-p^2} dp = \frac{1}{2}(1 - \mathcal{Erf}(x/\sqrt{4kt})),$$

where we used  $\int_{-\infty}^{-a} = \int_{-\infty}^0 - \int_{-a}^0$  and  $\int_{-a}^0 e^{-p^2} dp = \int_0^a e^{-p^2} dp$ , as  $e^{-p^2}$  is an even function. Hence we get

$$V(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}] \phi(y) dy + h'(t) \mathcal{Erf}(x/\sqrt{4kt}).$$

Substituting back, we obtain the solution

$$v(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}] \phi(y) dy + h'(t) \mathcal{Erf}(x/\sqrt{4kt}) + h(t).$$