## Section 2.3, Problem 4

For this problem it is necessary to use the *strong* maximum principle which is mentioned p. 41. This simply says that if $u$ is not constant, then one can replace $\leqslant$ in the maximum principle by $<$ in the interior. That is, if $u$ is not constant then its value at any point strictly inside the rectangle is strictly less than its maximum value on the bottom or two sides of that rectangle.

a) The maximum of $u(x,t)$ on the sides of any rectangle is always zero by the boundary conditions. A little calculus shows that $u(x,0)$ is maximized at $x_0 = \frac{1}{2}$, and $u(x_0,0) = 1$. Since $u$ is not constant (initially), by the strict maximum principle it must be that:

$$u(x,t) \ < \ 1 \ , \qquad\qquad 0 \ < \ t \ .$$

Furthermore, one sees that for $w = -u$ the maximum on the sides or bottom of any rectangle is zero. Thus:

$$-u(x,t) = w(x,t) \ < \ 0 \ , \qquad\qquad 0 \ < \ t \ .$$

But this last line is the same as:

$$0 \ < \ u(x,t) \ , \qquad\qquad 0 \ < \ t \ .$$

b) This follows from defining $w(x,t) = u(1-x,t)$. Based on the properties of $u$, its easy to see that:

$$\begin{aligned} \partial_t w \ &= \ \partial_x^2 w \ , \\ w(0,t) \ &= \ w(1,t) \ = \ 0 \ , \\ w(x,0) \ &= \ u(x,0) \ . \end{aligned}$$

Notice that the lat line above follows from the reflection symmetry of the graph of the function $u(x,0) = 4x(1-x)$ about the line $x = \frac{1}{2}$ (its just a concave down parabola with $x_0 = \frac{1}{2}$ as its max). From the uniqueness theorem for solutions to the heat equation, we see that we must have for all $(x,t)$ with $0 \leqslant t$:

$$u(x,t) \ \equiv \ w(x,t) \ .$$

But this is just the same things as saying that $u(x,t) = u(1-x,t)$.

## Section 2.3, Problem 8

This is a direct calculation using the boundary conditions. We compute that:

$$\begin{aligned}
\frac{dE}{dt}(t) &= \frac{d}{dt}\frac{1}{2}\int_0^\ell u^2(x,t)\ dx\ , \\
&= \int_0^\ell (u_t u)(x,t)\ dx\ , \\
&= k\int_0^\ell (u_{xx}u)(x,t)\ dx\ , \\
&= k(u_x u)(x,t)|_0^\ell - k\int_0^\ell (u_x)^2(x,t)\ dx\ , \\
&= k\big(-a_\ell u^2(\ell,t) - a_0 u^2(0,t)\big) - k\int_0^\ell (u_x)^2(x,t)\ dx\ .
\end{aligned}$$

The condition $0 < a_0, a_\ell$ now guarantees that this last line is non-positive, because all of the other terms involve squares and therefore have a sign (i.e. $0 \leqslant (\cdot)^2$). Thus:

$$\frac{dE}{dt}(t) \leqslant 0\ .$$

## Section 2.4, Problem 1

We plug the given initial function $\phi(x)$ into the general formula for the solution of the diffusion/heat equation. This gives us

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-l}^{l} e^{-(x-y)^2/4kt}dy.$$

Substituting $p = (y-x)/\sqrt{4kt}$, this integral can be rewritten as $Erf((l-x)/\sqrt{4kt}) - Erf((-l-x)/\sqrt{4kt})$.

## Section 2.4, Problem 7

This is a direct substitution. We have:

$$\begin{aligned}
\int_{-\infty}^{\infty} S(x,t)\ dx &= \int_{-\infty}^{\infty} e^{-x^2/4kt}\ \frac{dx}{\sqrt{4\pi kt}}, \\
&= \int_{-\infty}^{\infty} e^{-s^2}\ \frac{ds}{\sqrt{\pi}}\ , \\
&= 1\ .
\end{aligned}$$

The second to last line follows from the change of variables: $s = \frac{x}{\sqrt{4kt}}$.

## Section 2.4, Problem 18

Suppose that $u$ obeys the convection equation $u_t - ku_{xx} + Vu_x = 0$. Then define the new function $w(x,t) = u(y,t) = u(x+Vt,t)$. By the chain rule we have that:

$$w_t \;=\; u_t(y,t) + Vu_x(y,t)\ .$$

Also, its easy to see that $w_{xx}(x,t) = u_{xx}(y,t)$. Therefore, its easy to compute that:

$$w_t - kw_{xx} \;=\; u_t(y,t) + Vu_x(y,t) - ku_{xx}(y,t) \;=\; 0\ .$$

Furthermore, we have that $w(x,0) = u(x,0) = \phi(x)$. Therefore, by the explicit formula we have:

$$w(x,t) \;=\; \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\ dy\ .$$

But we also have the inverse relation $u(x,t) = w(x - Vt, t)$. Thus in fact:

$$\begin{aligned} u(x,t) \;&=\; \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-Vt-y)^2/4kt}\phi(y)\ dy\ , \\ &=\; \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y-Vt)\ dy\ . \end{aligned}$$