## 1. Section 1.1, Problem 3

Recall that an equation is *linear homogeneous* if you can add two solutions and get another. A good way to check for this is simply to see wether or not the equation involves any multiplications or non-linear functions of the solution (or its derivatives), and to check if zero is a solution. If zero is a solution, and there are no nonlinear terms, then the equation is linear homogeneous.

If the equation contains no non-linear terms, but has some kind of non-zero forcing term, then it is *linear inhomogeneous*. For example:

$$\partial_t u - k\partial_x^2 u \ = \ f(x,t) \ ,$$

is linear inhomogeneous.

*Nonlinear* equations involve multiplications or non-linear equations of the solution. A good example is:

$$\partial_t^2 u - \partial_x^2 u \ = \ \sin(u) \ .$$

(1) Linear Inhomogeneous. Here $-1$ (bringing it to the right hand side) is a "forcing term".
(2) Linear Homogeneous. You can add two solutions and zero is a solution.
(3) Nonlinear. The term $uu_x$ is nonlinear.
(4) Linear Inhomogeneous. Here $-x^2$ (bringing it to the right hand side) is a "forcing term".
(5) Linear Homogeneous. You can add two solutions and zero is a solution.
(6) Nonlinear. Many nonlinear terms here.
(7) Linear Homogeneous. Just a transport equation.
(8) Nonlinear. The term $\sqrt{1+u}$ is clearly non-linear.

## 2. Section 1.1, Problem 4

This is easy. The point is that the equation $\mathcal{L}u$ is linear. Therefore:

$$\begin{aligned} \mathcal{L}(u-w) \ &= \ \mathcal{L}u - \mathcal{L}w \ , \\ &= \ g - g \ \ = \ 0 \ . \end{aligned}$$

## 3. Section 1.2, Problem 1

Here the characteristic equation is (in $(t,x)$ coordinates):

$$\frac{dx}{dt} \ = \ \frac{3}{2} \ ,$$

with general solution $x = \frac{3}{2}t + C$. Thus, the general solution $u(x,t)$ looks like:

$$u(x,t) \ = \ F(x - \frac{3}{2}t) \ .$$

If when $t = 0$ we want $u(x,0) = \sin(x)$, then we need to choose $F(\xi) = \sin(\xi)$. Therefore, our specific solution is just:

$$u(x,t) \ = \ \sin(x - \frac{3}{2}t) \ .$$

## Section 1.2, Problem 3

We'll skip the drawings here. The characteristic ODE is:

$$\frac{dy}{dx} \ = \ \frac{1}{1+x^2} \ ,$$

to which the general solution is $y = \arctan(x) + C$. Thus, the general solution is:

$$u(x,y) \ = \ F(y - \arctan(x)) \ .$$

The interesting thing here is that the characteristics get trapped in horizontal strips if width $\pi$, because of the bounds $-\frac{\pi}{2} \leqslant \arctan(x) \leqslant \frac{\pi}{2}$. Also, the correct way to pose initial conditions for this problem would be at $x = 0$ but *not* at $y = 0$.

## Section 1.2, Problem 5

Here the characteristic equation is the ODE:

$$\frac{dy}{dx} \ = \ \frac{1}{1-x^2} \ ,$$

with general solution $y = \arcsin(x) + C$. Thus, the general solution is:

$$u(x,y) \ = \ F(y - \arcsin(x)) \ .$$

We can pick the branch of $\arcsin(x)$ such that $\arcsin(0) = 0$. Thus, if $u(0,y) = y$ we must have $F(\xi) = \xi$. Thus, our specific solution is:

$$u(x,y) \ = \ y - \arcsin(x) \ .$$

## Section 1.3, Problem 1

For these derivations we'll be a bit sketchy. Recall that the equation for the string *without* friction was derived from the linearized balance law ($F = ma$) on each interval $[a,b]$ ($a$ is not the acceleration here):

$$F \ = \ Tu_x|_a^b \ = \ \int_a^b \rho u_{tt} dx \ = \ ma \ .$$

Now, the point to the friction is that its a force which is proportional to the velocity, and goes in the *opposite* direction of motion (the faster the motion the more the friction). Therefore, we need to add a term like:

$$F_{friction} \ = \ -\int_a^b \gamma u_t dx \ ,$$

where $\gamma > 0$ is a constant with the correct dimensions that is meant to give the amount of friction per unit length per unit velocity. Thus, our new balance law looks like:

$$F_{total} \ = \ F_{int} + F_{friction} \ = \ Tu_x|_a^b - \int_a^b \gamma u_t dx \ = \ \int_a^b \rho u_{tt} dx \ = \ ma \ .$$

Here $F_{int}$ is the old "internal" force coming from tension in the string. Combining all of these terms on one side, and using the fundamental theorem of calculus as before, we arrive at the balance law:

$$\int_a^b (\rho u_{tt} - Tu_{xx} + \gamma u_t) dx \ = \ 0.$$

Since this must be true for every interval $[a, b]$ in the string length, we must have:

$$\rho u_{tt} - Tu_{xx} + \gamma u_t \ = \ 0 \ .$$

Setting $r = \rho^{-1}\gamma$ and $c^2 = \rho^{-1}T$ as usual, this may be written as:

$$u_{tt} + ru_t - c^2 u_{xx} \ = \ 0 \ .$$

## 4. Section 1.3, Problem 3

This is just a modification of the heat equation on a line segment. Recall that in the usual situation, the heat flow was computed via the balance law:

$$\frac{d}{dt}H(t) \ = \ \frac{d}{dt}\int_a^b u \ dx \ = \ ku_x|_a^b \ = \ \{k \text{ times flux across boundary}\} \ .$$

In this situation, there will also be a change in the heat across the segment $[a, b]$ due to heat exchange with the surrounding medium (at temp $T_0$). This heat change per unit time is computed as follows:

$$\{\text{heat exchange on } [a, b]\} \ = \ -\int_a^b \gamma(u - T_0) \ dx \ .$$

That is, the heat exchange is proportional (here $\gamma > 0$) to the total temperature difference on this interval. The constant $\gamma$ is computed in terms of the material, and we won't bother with that here. The new balance law is:

$$\frac{d}{dt}\int_a^b u \ dx \ = \ ku_x|_a^b - \int_a^b \gamma(u - T_0) \ dx \ .$$

Combining all of these terms on one side, and using the fundamental theorem of calculus as before, we have the zero balance law:

$$\int_a^b \left(u_t - ku_{xx} + \gamma(u - T_0)\right)\, dx\ .$$

Since $[a, b]$ is arbitrary, we have that $u$ obeys the equation:

$$u_t \;=\; ku_{xx} - \gamma(u - T_0)\ .$$

## 5. Section 1.3, Problem 9

This is a direct application of the divergence theorem. One readily computes that:

$$\nabla \cdot \mathbf{F} \;=\; 5r^2\ ,$$

therefor its integral is:

$$\begin{aligned} \iiint_{r\leqslant a} \nabla \cdot \mathbf{F} d\mathbf{x} \;&=\; \int_0^{2\pi}\int_0^{\pi}\int_0^a 5r^2\, \sin(\phi) r^2 dr d\phi d\theta\ , \\ &=\; 4\pi \int_0^a 5r^4 dr\ , \\ &=\; 4\pi a^5\ . \end{aligned}$$

Also, on the boundary at $r = a$ one computes that the outward unit normal is $\mathbf{n} = \widehat{\mathbf{x}} = (x/r, y/r, z/r)$. Furthermore, you will recall that the area element on a sphere radius $r = a$ is $dS = a^2 \sin(\phi) d\phi d\theta$. Thus, we have that:

$$\begin{aligned} \iint_{r=a} \widehat{\mathbf{x}} \cdot \mathbf{F}\ dS \;&=\; \int_0^{2\pi}\int_0^{\pi} a^3\, a^2 \sin(\phi) d\phi d\theta\ , \\ &=\; 4\pi a^5\ . \end{aligned}$$

Of course this is what we had from before.

## 6. Section 1.4, Problem 1

We have $u_{xx}(x, 0) = 2$. Hence we can try $u(x, t) = x^2 + f(t)$ with $u_t(x, t) = f'(t)$. Hence $2 = u_x x(x, 0) = f'(0)$. Check that $u(x, t) = x^2 + 2t$ is a solution. *Remark*: For the general solution, see Section 2.4, Problem 9.

## 7. Section 1.4, Problem 3

With insulating boundary conditions, one has that $\frac{dH}{dt} = 0$ where $H(t)$ is the total heat content:

$$H(t) \;=\; \int_\Omega u(x, t)\ dx\ .$$

It is a fact that in this kind of situation, the function $u(x,t)$ will become constant as $t \to \infty$ (we will actually prove this later for the simple case of a line segment $\Omega = [a,b]$). Therefore, let:

$$T_0 \;=\; \lim_{t\to\infty} u(x,t) \;,$$

be the steady state temperature. Thus, we have that:

$$\lim_{t\to\infty} H(t) \;=\; \int_\Omega T_0 \; dx \;=\; T_0 \cdot |\Omega| \;.$$

Here $|\Omega|$ denotes the volume of the domain $\Omega$. But $H(t)$ is a constant, so if we set $f(x) = u(x,0)$ to be the initial temperature distribution, then we have:

$$H(t) \;=\; \int_\Omega f(x) \; dx \;.$$

Therefore, we must have that:

$$T_0 \;=\; \frac{1}{|\Omega|}\int_\Omega f(x) \; dx \;.$$

That is, the steady state temperature is simply the average of the initial temperature distribution. Put another way, the heat flow unction $u(x,t)$ converges to its (initial) average as $t \to \infty$.

## 8. Section 2.1: Problem 2

Just plug into the solution formula for initial value problems of the wave equation to get

$$u(x,t) = \frac{1}{2}(\log(1+(x+ct)^2) + (\log(1+(x-ct)^2) + \frac{1}{2c}\int_{x-ct}^{x+ct} 4 + sds =$$

$$= \frac{1}{2}(\log(1+(x+ct)^2) + \log(1+(x-ct)^2)) + 4t + tx$$

.

## 9. Section 2.1: Problem 5

Sorry, my computer graphing is rudimentary. Here are just the major points for finding the solution. I did it a little bit more general. If you do not like this, just plug in one of the values of $t$ as indicated in the book, and go through the case described below for which it applies. As indicated in the book, $u(x,t)$ is equal to the length of the intersection of the intervals $[-a,a] \cap [x-ct, x+ct]$, divided by $2c$. We observe from this the following:

(a) $u(x,t) = 0$ if the intersection is empty, i.e. if $a < x - ct$ or $-a > x + ct$, i.e. if $|x| > |a + ct|$.

(b) The length of the intersection can be at most equal to the length of $[-a,a] = 2a$. Hence $u(x,t) \leq 2a/2c = a/c$.

(c) The maximum value in (b) can only be obtained, if $[-a, a] \subset [x - ct, x + ct]$. The second interval has length $2ct$, hence this is possible only if $2ct \geq 2a$, i.e. if $t \geq a/c$ and if $x - ct \leq -a$ and $x + ct \geq a$. Hence we have $u(x, t) = a/c$ if $a - ct \leq x \leq ct - a$ (if $t > c/a$).

(d) If $t > c/a$, we have seen that $u(x, t) = 0$ for $|x| > |a + ct|$, by (a), and $u(x, t) = a/c$ if $|x| < |ct - a|$, by (c). Sketch this part of the graph! It remains to show that the missing bits of the graph are given by straight lines. Check for yourself that for $|ct - a| < |x| < |ct + a|$ the length of $[-a, a] \cap [x - ct, x + ct]$ is equal to $|a + ct| - |x|$ (consider the cases $x > 0$ and $x < 0$ separately). Conclude that for $ct - a < x < ct + a$ we have $u(x, t) = (-x + a + ct)/2c$.

(e) Show that for $t < c/a$ the maxium value of $u(x, t)$ is equal to $t$ (this happens whenever $[x - ct, x + ct] \subset [a, a]$). Determine as before for which values $x$ the maximum value can be obtained, and for which values of $x$ the function $u(x, t) = 0$, in this case.

If you want to see sketches, ask me in class or the TA in section.