MATH 110 MIDTERM WINTER 2010

Please justify all your steps!

1. (a) Find the general solution of the transport equation $u_x + 2xe^y u_y = 0$.
   (b) Find the particular solution of (a) satisfying $u(x, 0) = x^2$.
   (c) *Extra Credit*: Find the general solution of $u_x + 2xe^y u_y = x$ (*Hint*: Consider solutions of the form $u(x, t) = h(x) + k(y)$).

2. Consider the wave equation $u_{tt} = u_{xx}$ for $-\infty < x < \infty$ with the initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = \psi(x)$, where

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

   (a) Calculate $u(0, t)$ explicitly for all $t > 0$ for the solution $u(x, t)$ of the just stated wave equation.
   (b) Show that $\lim_{t\to\infty} u(x, t) = 1$ for all x, $-\infty < x < \infty$.
   (c) What is $\lim_{t\to\infty} E(t)$, where $E(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_x(x, t)^2 + u_t(x, t)^2 dx$?

3. Consider the heat equation $u_t = ku_{xx}$ with the initial condition $u(x, 0) = e^{-x^2}$ for $-\infty < x < \infty$.
   (a) Write down the solution in terms of an integral. Do NOT solve the integral.
   (b) Prove that the integral in (a) is equal to $(4kt + 1)^{-1/2} e^{-x^2/(4kt+1)}$ (*Hint* : There is a shorter way than actually calculating the integral).
   (c) *Extra Credit*: Now consider the same initial value problem as in (a) on the half line $0 \leq x < \infty$. Solve it either with the boundary condition $u(0, t) = 0$ or with the boundary condition $u_x(0, t) = 0$ for all $t > 0$ in each case (One case is easier than the other one).