MATH 110 MIDTERM SOLUTIONS

1. We obtain the characteristic curves by solving the differential equation

$$y' = \frac{y}{x}.$$

Separating variables, we obtain $\int \frac{1}{y}\ dy = \int \frac{1}{x}\ dx$, from which we deduce $\log y = \log x + \tilde{c}$. Exponentiating and setting $c = e^{\tilde{c}}$, we see that the characteristic curves are given by

$$y = cx,$$

i.e. lines going through the origin. Moreover, we obtain $c = y/x$. Hence the general solution of the transport equation is given by

$$u(x, y) = f(c) = f(y/x),$$

for f a differentiable function. To determine f in our case, we observe that

$$x^2 = f(x, 1) = f(1/x).$$

Hence $f(c) = 1/c^2$ in our case, and hence the solution of our problem is given by

$$u(x, y) = x^2/y^2.$$

2. (a) The general solution of the wave equation for $c = 1$ is given by

$$u(x, t) = \frac{1}{2}[\phi(x + t) + \phi(x - t) + \ \frac{1}{2}\int_{x-t}^{x+t} \psi(y)\ dy.$$

We then obtain for $u(1, 3/2)$ the value

$$u(1, 3/2) = \frac{1}{2}[\phi(5/2) + \phi(-1/2)] + \ \frac{1}{2}\int_{-1/2}^{5/2} \psi(y)\ dy\ =$$

$$= \ \frac{1}{2}[0 - 1/2 + \int_{-1/2}^{1} 1\ dy]\ = \ \frac{1}{2}.$$

(b) We can use the same formula for the solution $v(x, t)$ with Neumann boundary condition, provided we substitute ϕ and ψ by ϕ_{ev} and ψ_{ev} respectively. We see that $\psi = \psi_{ev}$ is even, while $\phi_{ev}(x) = |\phi(x)|$ in our case. Plugging this into the solution formula for $v(1, 3/2)$, we see that

$$v(1, 3/2) = \frac{1}{2}[0 + 1/2 + \int_{-1/2}^{1} 1\ dy]\ = \ 1.$$

Remark : Originally, you were given the problem for calculating the functions $u(1, 1/2)$ and $v(1, 1/2)$. In this case, both solutions were equal to $1/2$.

3. Setting $\lambda = \beta^2$, we obtain for the differential equation $X'' + \beta^2 X = 0$ the general solution

$$X(x) = A\cos\beta x + B\sin\beta x.$$

As $u(x,t) = X(x)T(t)$, we obtain

$$0 \;=\; u(0,t) \;=\; X(0)T(t) \;=\; (A\cos 0 + B\sin 0)T(t) \quad \text{for all } t.$$

We can assume that $T(t)$ is not the zero function (as otherwise $u(x,t)$ would be zero for all x, t, which is not an interesting solution). Hence we have

$$0 \;=\; (A\cos 0 + B\sin 0) = A,$$

i.e. we can assume $X(x) = \sin\beta x$. The second boundary condition $u_x(\ell, t) = 0$ similarly translates to

$$0 = X'(\ell) = \beta\cos\beta\ell.$$

As the zeros of cos are at $(n + \frac{1}{2})\pi$, we obtain

$$\beta\ell = (n + \frac{1}{2})\pi,$$

from which we can calculate β as it is claimed.

(b) It follows from $u(x,t) = X(x)T(t)$ and the separation of variables that

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda = -\beta^2 = -[(n + \frac{1}{2})\pi/\ell]^2.$$

Hence we get the differential equation $T' = -k[(n + \frac{1}{2})\pi/\ell]^2 T$ which has the solution

$$T(t) = e^{-k[(n+\frac{1}{2})\pi/\ell]^2 t}.$$

We obtain the series solution

$$\sum_{n=0}^{\infty} A_n \sin[(n + \frac{1}{2})\pi x/\ell]\; e^{-k[(n+\frac{1}{2})\pi/\ell]^2 t}.$$

(c) (sketch) As $|\sin y| \le 1$ for all y, we have

$$|\sin[(n + \frac{1}{2})\pi x/\ell]\; e^{-k[(n+\frac{1}{2})\pi/\ell]^2 t}| \le |e^{-k[(n+\frac{1}{2})\pi/\ell]^2 t}| \;\to\; 0, \quad \text{for } t \to \infty.$$

Then also finite sums of such functions will go to 0 if t goes to infinity. This solves the problem in the special case if only A_0 and A_1 are not equal to 0. The case for a general series is more complicated to prove. But also in this case the solution will go to 0 if $t \to \infty$.