Math 110A Midterm Solutions

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1. (a) The characteristic curves satisfy ODE

$$\frac{dy}{dx} = 2xe^y$$

Separating variables and integrating, we obtain

$$\int e^{-y}\,dy = \int 2x\,dx$$

Then $-e^{-y} = x^2 + C$ or $-C = x^2 + e^{-y}$. Thus, the general solution to our equation is $u(x, y) = f(x^2 + e^{-y})$ (for arbitrary differentiable f).

(b) We have $u(x, 0) = f(x^2 + 1)$, and we want this to equal x^2. Thus, we choose $f(s) = s - 1$, and our particular solution is then $u(x, y) = x^2 + e^{-y} - 1$.

(c) Suppose $u(x, y) = h(x) + k(y)$ and plug in to the equation. This gives

$$h'(x) + 2xe^y k'(y) = x$$

Rearranging terms gives us

$$\frac{h'(x) - x}{2x} = -e^y k'(y)$$

However, the left hand side depends only on x whereas the right hand side depends only on y. Thus, both must be equal to some constant λ. This in turns gives us the pair of ODEs

$$h'(x) = (2\lambda + 1)x$$
$$k'(y) = -\lambda e^{-y}$$

with particular solutions $h(x) = (\lambda + \frac{1}{2})x^2 = \lambda x^2 + \frac{1}{2}x^2$ and $k(y) = \lambda e^{-y}$. Thus, a particular solution to our equation is $\frac{1}{2}x^2 + \lambda(x^2 + e^{-y})$. We conclude that the general solution to the inhomogeneous equation is

$$u(x, y) = \tfrac{1}{2}x^2 + f(x^2 + e^{-y})$$

where f is any differentiable function (since the $\lambda(x^2 + e^{-y})$ term can be absorbed into f).

2. (a) We have $u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \psi(s)\,ds$, so it follows that $u(0,t) = \frac{1}{2}\int_{-t}^{t} \psi(s)\,ds$. Note that the value of the integral is the length of the intersection of $(-t,t)$ and $(-1,1)$. So if $0 < t \le 1$, we get $\frac{1}{2}(t-(-t)) = t$, but if $t > 1$, we get $\frac{1}{2}(1-(-1)) = 1$. Summarizing, we have

$$u(0,t) = \begin{cases} t & 0 < t \le 1 \\ 1 & t > 1 \end{cases}$$

(b) Fix any $x \in (-\infty,\infty)$. Note that $u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \psi(s)\,ds$. The value of this integral is equal to the length of the intersection of $(x-t, x+t)$ and $(-1,1)$. However, for t sufficiently large (to be exact: $t > \max\{x+1, -x+1\}$) the interval $(x-t, x+t)$ contains $(-1,1)$ so we get

$$\lim_{t\to\infty} u(x,t) = \frac{1}{2}(1-(-1)) = 1$$

(c) We know that the energy $E(t)$ is conserved for the wave equation. In particular,

$$\lim_{t\to\infty} E(t) = E(0) = \frac{1}{2}\int_{-\infty}^{\infty} u_x(x,0)^2 + u_t(x,0)^2\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \psi(x)^2\,dx = 1$$

3. (a)

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-y^2} e^{-\frac{(x-y)^2}{4kt}}\,dy$$

(b) Define

$$v(x,t) = \frac{1}{\sqrt{4kt+1}} e^{-\frac{x^2}{4kt+1}}$$

Taking derivatives shows that $v_t = kv_{xx}$ so $v(x,t)$ satisfies the heat equation. Furthermore, we have $v(x,0) = e^{-x^2}$. Thus, by uniqueness, we have $v(x,t) = u(x,t)$ from (a) which is the desired result.

(c) The easier case is the one with the Neumann condition $u_x(0,t) = 0$. To solve this, we would have used even reflection. However, our initial data is already even when considered as a function on the whole line. Therefore our solution formula remains unchanged and we get

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-y^2} e^{-\frac{(x-y)^2}{4kt}}\,dy \qquad \text{for } x \in [0,\infty)$$