Math 110A Homework Solutions

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6.3.3 Solve $u_{xx} + u_{yy} = 0$ in the disk $\{r < a\}$ with the boundary condition

$$u = \sin^3\theta \qquad \text{on } r = a$$

Solution

We use the identity $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$ to write

$$u(a,\theta) = \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta)$$

The solution to our problem (as derived in the text) is given by

$$u(r,\theta) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n \left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)$$

Thus, plugging in $r = a$ gives us

$$u(a,\theta) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n A_n\cos(n\theta) + a^n B_n\sin(n\theta)$$

By inspection, we see that we will get the right function on the boundary if $A_0 = 0$ for all n and $B_1 = \frac{3}{4a}$, $B_2 = 0$, $B_3 = -\frac{1}{4a^3}$, $B_n = 0$ for $n \geq 4$. This in turn gives

$$u(r,\theta) = \frac{3}{4a}r\sin\theta - \frac{1}{4a^3}r^3\sin(3\theta)$$

This is the unique solution to our problem.

6.4.3 The solution derived in the text is

$$u(r,\theta) = \frac{1}{2}\left(C_0 + D_0\log r\right) + \sum_{n=1}^{\infty}\left(C_n r^n + D_n r^{-n}\right)\cos(n\theta) + \left(A_n r^n + B_n r^{-n}\right)\sin(n\theta)$$

Setting $r = a$ and then $r = b$, we obtain (respectively)

$$g(\theta) = \frac{1}{2}\left(C_0 + D_0\log a\right) + \sum_{n=1}^{\infty}\left(C_n a^n + D_n a^{-n}\right)\cos(n\theta) + \left(A_n a^n + B_n a^{-n}\right)\sin(n\theta)$$

$$h(\theta) = \frac{1}{2}\left(C_0 + D_0 \log b\right) + \sum_{n=1}^{\infty} \left(C_n b^n + D_n b^{-n}\right)\cos(n\theta) + \left(A_n b^n + B_n b^{-n}\right)\sin(n\theta)$$

We notice that the right hand sides are simply Fourier series expansions and conclude that the following equalities must hold:

$$\begin{aligned}
\frac{1}{2}\left(C_0 + D_0 \log a\right) &= \frac{1}{\pi}\int_0^{2\pi} g(\phi)\, d\phi \\
\frac{1}{2}\left(C_0 + D_0 \log b\right) &= \frac{1}{\pi}\int_0^{2\pi} h(\phi)\, d\phi \\
C_n a^n + D_n a^{-n} &= \frac{1}{\pi}\int_0^{2\pi} g(\phi)\cos(n\phi)\, d\phi \\
C_n b^n + D_n b^{-n} &= \frac{1}{\pi}\int_0^{2\pi} h(\phi)\cos(n\phi)\, d\phi \\
A_n a^n + B_n a^{-n} &= \frac{1}{\pi}\int_0^{2\pi} g(\phi)\sin(n\phi)\, d\phi \\
A_n b^n + B_n b^{-n} &= \frac{1}{\pi}\int_0^{2\pi} h(\phi)\sin(n\phi)\, d\phi
\end{aligned}$$

Solving this system yields:

$$\begin{aligned}
C_0 &= \frac{2}{\pi(1 - \frac{\log b}{\log a})}\int_0^{2\pi} \left(h(\phi) - \tfrac{\log b}{\log a} g(\phi)\right)\, d\phi \\
D_0 &= \frac{2}{\pi \log(\frac{a}{b})}\int_0^{2\pi} \left(g(\phi) - h(\phi)\right)\, d\phi \\
D_n &= \frac{1}{\pi(a^{-2n} - b^{-2n})}\int_0^{2\pi} \left[a^{-n} g(\phi) - b^{-n} h(\phi)\right]\cos(n\phi)\, d\phi \\
C_n &= \frac{1}{\pi(a^{2n} - b^{2n})}\int_0^{2\pi} \left[a^{n} g(\phi) - b^{n} h(\phi)\right]\cos(n\phi)\, d\phi \\
B_n &= \frac{1}{\pi(a^{-2n} - b^{-2n})}\int_0^{2\pi} \left[a^{-n} g(\phi) - b^{-n} h(\phi)\right]\sin(n\phi)\, d\phi \\
A_n &= \frac{1}{\pi(a^{2n} - b^{2n})}\int_0^{2\pi} \left[a^{n} g(\phi) - b^{n} h(\phi)\right]\sin(n\phi)\, d\phi
\end{aligned}$$

6.4.5 (a) We are to solve the equation

$$u_{xx} + u_{yy} = 0$$
$$u_r(2, \theta) = 0$$
$$u(1, \theta) = \sin^2 \theta$$

in the annulus $\{1 < r < 2\}$.

We have separated solutions $C_0 + D_0 \log r$, $Cr^n + Dr^{-n}$, and $A\cos(n\theta) + B\sin(n\theta)$. Posing the homogeneous boundary condition $u_r(2, \theta) = 0$ allows us to conclude that $D_0 = 0$ and $nC2^{n-1} - nD2^{-n-1} = 0$ or rather $D = 4^n C$. We choose $C = 1$ with no loss of information. Thus

$$u(r, \theta) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} \left[A_n(r^n + 4^n r^{-n})\cos(n\theta) + B_n(r^n + 4^n r^{-n})\sin(n\theta)\right]$$

Now we simply require

$$\sin^2 \theta = \tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} [A_n(1 + 4^n)\cos(n\theta) + B_n(1 + 4^n)\sin(n\theta)]$$

So we need $A_0 = 1$, $A_2 = -\frac{1}{34}$, and the rest of the coefficients are 0. Therefore our solution is given by

$$u(r, \theta) = \tfrac{1}{2} - \tfrac{1}{34}(r^2 + 16r^{-2})\cos(2\theta)$$

(b) This time we impose the boundary condition $u(2, \theta) = 0$, which leads (in a similar manner) to the solution

$$u(r, \theta) = \tfrac{1}{2}A_0(\log r - \log 2) + \sum_{n=1}^{\infty} \left[A_n(r^n - 4^n r^{-n})\cos(n\theta) + B_n(r^n - 4^n r^{-n})\sin(n\theta)\right]$$

So we need $A_0 = -\frac{1}{\log 2}$, $A_2 = -\frac{1}{30}$, and the rest are 0. This gives

$$u(r, \theta) = \tfrac{1}{2} - \tfrac{\log r}{2\log 2} - \tfrac{1}{30}(r^2 - 16r^{-2})\cos(2\theta)$$