Math 110A Homework Solutions

C.T. Wildman

4.3.4 Suppose $a_0, a_l < 0$ and $-(a_0+a_l) < a_0 a_l l$. Show that there are two negative eigenvalues for the problem $X'' = -\lambda X$.

Solution:

Following the hint, we define $y(\gamma) = \frac{-(a_0+a_l)\gamma}{\gamma^2+a_0a_l}$. Solving the equation $y'(\gamma) = 0$ and applying the first derivative test, we see that y has a single maximum at $\gamma = \sqrt{a_0 a_l}$. We also observe that $y'(0) = \frac{-(a_0+a_l)}{a_0 a_l}$. By assumption, this quantity is (strictly) less than l. What this means is that the function $y(\gamma)$ is less steep at the origin than $\tanh(\gamma l)$ (which has slope l at 0).

The next step is to show that $y(\gamma)$ intersects $y = 1$ at least once. Therefore we consider the equation $y(\gamma) = 1$ and obtain

$$\gamma^2 + (a_0 + a_l)\gamma + a_0 a_l = 0$$

which has 2 solutions if $a_0 \neq a_l$ and 1 solution if $a_0 = a_l$. In either case, we know that $\tanh(\gamma l) < 1$ for all γ, so since $\lim_{\gamma\to\infty} y(\gamma) = 0$ and $\lim_{\gamma\to\infty} \tanh(\gamma l) = 1$, we see that $y(\gamma)$ must intersect $\tanh(\gamma l)$ twice: $y(\gamma)$ is less steep at 0, but it touches $y = 1$ and then comes back down to 0, whereas $\tanh(\gamma l)$ is steeper at 0 but increases to 1.

Thus, there are two solutions to $y(\gamma) = \tanh(\gamma l)$, i.e. two negative eigenvalues.