MATH 102 MIDTERM Spring 2008

Please justify all your steps!

1. Let $A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 9 & 7 \\ 3 & 6 & 11 & 8 \end{bmatrix}$.
(a) Calculate bases for the null space and for the column space of A.
(b) Calculate the general solution of $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (2, 5, 5)^T$.
(c) Describe the column space of A via linear equation(s) and find a vector $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

2. Let A be a 4×3 matrix whose columns are linearly independent.
(a) What is the rank of A and what is the dimension of its null space?
(b) The equation $A\mathbf{x} = \mathbf{b}$ always has at least one solution, for any vector $\mathbf{b} \in \mathbf{R}^4$. True or false? Why or why not?
(c) The equation $A\mathbf{x} = \mathbf{b}$ has at most one solution, for any vector $\mathbf{b} \in \mathbf{R}^4$. True or false? Why or why not?

3. Let $\mathbf{a}_1 = (1, 1, 1, 1)^T$, $\mathbf{a}_2 = (3, 1, 1, 3)^T$ and $\mathbf{a}_4 = (4, 0, -2, 2)^T$.
(a) Find an orthogonal basis for span$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.
(b) Let A be the 4×3 matrix with column vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. Find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.
(c) Let $\mathbf{b} = (6, 0, 2, 0)^T$. Find the projection of $\mathbf{b}$ onto the span of $\mathbf{a}_1$ and $\mathbf{a}_2$.
(d) Let $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ be the orthogonal basis obtained in (a). Then it can be extended to an orthogonal basis for $\mathbf{R}^4$ by taking a suitable vector $\mathbf{q}_4$ from the null space of A^T. True or false? Why or why not? If true, how many such vectors would there be?

4. Let A be a 3 x 3 matrix satisfying $A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ and $A \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Compute $A \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.