

① a) DO Gaussian elimination on A:

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 9 & 7 \\ 3 & 6 & 11 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\uparrow \quad \uparrow$ basis for $\text{col}(A)$. $\uparrow \quad \uparrow$ pivot cols.

$$\text{Col}(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 11 \end{pmatrix} \right\}$$

Find $\text{Nul}(A)$ by solving homogeneous equation:

$$x_1 + 2x_2 + 4x_3 + 3x_4 = 0$$

$$x_3 + x_4 = 0 \Rightarrow x_4 = -x_3$$

$$x_1 + 2x_2 + x_3 = 0$$

$$\Rightarrow x_3 = -x_1 - 2x_2$$

$$x_4 = x_1 + 2x_2$$

$$\text{Nul}(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ -x_1 - 2x_2 \\ x_1 + 2x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ -2 \\ 2 \end{pmatrix} \right\}$$

$$\Rightarrow \text{Nul}(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ 2 \end{pmatrix} \right\}$$

b) $\left(\begin{array}{cccc|c} 1 & 2 & 4 & 3 & 2 \\ 2 & 4 & 9 & 7 & 5 \\ 3 & 6 & 11 & 8 & 5 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 4 & 3 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \Rightarrow$

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + 3x_4 &= 2 \\ x_3 + x_4 &= 1 \\ \Rightarrow x_4 &= 1 - x_3 \\ \Rightarrow x_1 &= 2 - 2x_2 - x_3 \end{aligned}$$

solutions to the system are

$$\left\{ \begin{pmatrix} 2 - 2x_2 - x_3 \\ x_2 \\ x_3 \\ 1 - x_3 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

c) $\text{Col } A = \{b \in \mathbb{R}^3 \text{ s.t. } AX=b \text{ is consistent}\}$

#1

Let $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and row-reduce:

$$\left(\begin{array}{cccc|c} 1 & 2 & 4 & 3 & b_1 \\ 2 & 4 & 8 & 7 & b_2 \\ 3 & 6 & 11 & 8 & b_3 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 4 & 3 & b_1 \\ 0 & 0 & 1 & 1 & b_2 - 2b_1 \\ 0 & 0 & -1 & -1 & b_3 - 3b_1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 4 & 3 & b_1 \\ 0 & 0 & 1 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - 3b_1 + b_2 - 2b_1 \end{array} \right)$$

So in order for $b \in \text{Col}(A)$, need

$$b_3 + b_2 - 5b_1 = 0$$

Any b for which this equation does not hold is a vector for which $AX=b$ has no solution.

Key Idea :

$b \in \text{Col}(A)$ if and only if $AX=b$ has a solution!

②

a) A has LI columns \Rightarrow $\boxed{\text{Rank}(A) = 3}$

$$\text{Rank}(A) + \dim(\text{Nul } A) = 3 \Rightarrow \boxed{\dim(\text{Nul } A) = 0}$$

b) $\boxed{\text{False}}$ since $\dim(\mathbb{R}^4) = 4 > \text{Rank}(A)$

(i.e., there are vectors in \mathbb{R}^4 that aren't in $\text{Col}(A)$).

c) $\boxed{\text{True}}$ because A has a trivial

nulspace (equivalently, because the columns of A are LI).

③

a) Apply Gram-Schmidt to $\{a_1, a_2, a_3\}$ and

$$\text{get } w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, w_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.$$

b) cols of Q are orthonormal basis for $\text{Col}(A)$. We already found an orthogonal basis in (a); we just need to normalize to get the columns of Q :

$$\|w_i\| = 2 \Rightarrow q_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, q_2 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, q_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{pmatrix}$$

$$Q = [q_1 \ q_2 \ q_3], \quad R = Q^T A = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

c) We already found an orthogonal basis $\{w_1, w_2\}$ for $\text{span}\{a_1, a_2\}$, so we can project b onto w_1 and w_2 and then add them.

Key Idea: You can only project onto an orthogonal basis!

$$\begin{aligned} \text{proj}_{\text{span}\{a_1, a_2\}} b &= \text{proj}_{w_1} b + \text{proj}_{w_2} b \\ &= \frac{(1\ 0\ 2\ 0) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}{(1\ 1\ 1\ 1) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{(1\ 0\ 2\ 0) \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}}{(1\ -1\ -1\ 1) \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 1 \\ 1 \\ 3 \end{pmatrix} \end{aligned}$$

d) True $\text{Nul}(A^T) = (\text{col } A)^\perp$, so any vector q_4 in $\text{Nul}(A^T)$ extends $\{q_1, q_2, q_3\}$ into an orthogonal basis for \mathbb{R}^4 .

④ $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, so by linearity of A ,

$$A \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 2A \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + A \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix}$$