Piecewise Riccati Equations and The Bounded Real Lemma *

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Abstract

This article concerns piecewise linear systems and determining if they meet given H^∞ performance specifications. Such problems occur in control of linear systems with saturation nonlinearities. While one could imagine many mathematically natural piecewise linear systems problems we took care to extract one which does correspond to control of saturated plants. We think many such problems will fall into the category treated here.

After a serious compromise (which makes our test for meeting specifications conservative), we arrive at a type of piecewise Riccati inequality. We show how these can be converted to matrix inequalities which are linear in the unknowns. Thus modern methods for optimizing such expressions can be used to obtain solutions.

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Section 1. INTRODUCTION AND PROBLEM STATEMENT

We consider a piecewise affine linear SISO system on state space R^n. It has two parts:

$$\begin{aligned} dv/dt &= \tilde{A}_1 v + \tilde{B}_1 W \\ out &= \tilde{C}_1 v + \tilde{D}_1 W \end{aligned} \tag{1.1a'}$$

when $cv \leq 1$ and

$$\begin{aligned} dv/dt &= \tilde{A}_2 v + \tilde{B}_2 W + a \\ out &= \tilde{C}_2 v + \tilde{D}_2 V + b \end{aligned} \tag{1.1b'}$$

when $cv \geq 1$. Here c is a row vector in R^n, so the set $\Delta := \{v : cv = 1\}$ is a hyperplane in R^n. Thus we have a linear system on each side of Δ and a transition from one linear system to another occurs at Δ. This article treats systems which are continuous across Δ; however they have a corner at Δ. That is the right hand side of $(1.1a')$ equals that of $(1.1b')$ for all v satisfying $cv = 1$.

For the system $(1.1b')$ to equal $(1.1a')$ on $cv = 1$ one can check that it must have the form

$$\begin{aligned} dv/dt &= (\tilde{A}_1 - ac)v + \tilde{B}_1 W + a \\ out &= (\tilde{C}_1 - bc)v + \tilde{D}_1 W + b \end{aligned} \tag{1.1b''}$$

The continuity assumption holds for systems with saturation of the type we are preparing to treat. Henceforth we suppress subscripts tildes etc. and write our system as

$$\begin{aligned} dv/dt &= Av + BW \\ out &= Cv + DW \end{aligned} \tag{1.1a}$$

when $cv \leq 1$ and

$$\begin{aligned} dv/dt &= (A - ac)v + BW + a \\ out &= (C - bc)v + DW + b \end{aligned} \tag{1.1b}$$

when $cv \geq 1$. We shall take $D = 0$ to simplify formulas although one could easily extend the results here to $D \neq 0$.

a. The Ideal Problem

The basic issue is to determine if the system (1.1a) (1.1b) is Input Output-dissipative (**IO-dissipative**). That is, initialize with $x_0 = 0$, then

$$\int_0^\infty ||W||^2 dt \geq \int_0^\infty ||out||^2 dt \tag{1.2}$$

for all input functions $W \in L^2[0,\infty]$. Moreover, the system must be asymptotically stable.

For linear systems, the most common test for dissipativity is the famous bounded real lemma (BRL) and various non-linear versions of this have appeared. We list recent publications of four groups of investigators [ISY], [BH], [J], [VS]. Since our system while continuous is not smooth, we do not expect smooth solutions to the Hamilton-Jacobi-Bellman equations produced by the BRL, but we must deal with weak solutions which possibly have corners. There is a well developed theory of such weak solutions and they have the name **viscosity solutions**. Preprints [BH] and [J] deal with this.

To apply results of those articles to the case at hand we associate to the system (1.1) a function H^{*W} : $R^n \times R^n \times R \longrightarrow R$ called the Hamiltonian defined [1] by

$$H^{*W}(v,p) = \begin{cases} H_1^{*W}(v,p) & if \quad cv \leq 1 \\ H_2^{*W}(v,p) & if \quad cv > 1 \end{cases} \tag{1.3a}$$

where

$$H_1^{*W}(v,p) = ||Cv||^2 + p^T[Av + BB^Tp/4] \tag{1.4}$$

$$H_2^{*W}(v,p) = ||[C-bc]v + b||^2 + \ p^T[(A-ac)v + BB^Tp/4 + a] \tag{1.5}$$

in the unknown $p = \nabla\varepsilon$. Note that for $v \in \Delta$ and all p we have

$$H_1^{*W}(v,p) = H_2^{*W}(v,p).$$

We shall be most interested in solutions ε to the first order partial differential inequality.

$$H^{*W}(v,p) \leq 0 \tag{1.6}$$

[1] The Hamiltonian H^{*W} is derived from a preHamiltonian H in a standard way which we describe The preHamiltonian for this problem is

$$H(v,p,W) = \begin{cases} H_1 & if \quad cv \leq 1 \\ H_2 & if \quad cv > 1 \end{cases}$$

where

$$H_1 = ||Cv||^2 - ||W||^2 + p^T(Av + BW)$$

$$H_2 = ||(C-bc)v + b||^2 - ||W||^2 + p^T[(A-ac)v + BW + a]$$

we call this the Hamiltonian of (1.1). Also denote by H^{*W}

$$H^{*W}(v,p) = \max_W H(v,p,W)$$

and observe that it can be computed explicitly since H is quadratic in W. The critical W is $B^Tp/2$, so we calculate that H^{*W} consists of the pieces H_1^{*W}, H_2^{*W} giving the formula (1.3) where $p \in \nabla^+\varepsilon(v)$ is the **superdifferential** of the function ε on R^n. Recall that the superdifferential $\nabla^+\varepsilon(v_0)$ of ε at the point v_0 is the set of p in R^n satisfying

$$\lim_{v \to v_0} \sup\{\varepsilon(v) - \varepsilon(v_0) - p^T(v - v_0)\}|v - v_0|^{-1} \leq 0,$$

an ε satisfying (1.6) is called a **viscosity subsolution** for $H^{*W}(v, \nabla\varepsilon(v)) \leq 0$. The results of [BH] and [J] imply that (1.1) is IO-dissipative if (resp. only if) there is a viscosity subsolution $\varepsilon > 0$ except at $v = 0$ where we have $\varepsilon(0) = 0$ (resp. $\varepsilon \geq 0$). Controllability of the system implies continuity of any viscosity subsolution ε. Thus solving (1.6) is the basic problem we would like to solve.

b. Compromised Problem

There is no evidence that the viscosity subsolutions we seek are at all nice. Since in many problems the state space where one would like to work is 50 or so dimensional, even if we managed to find the general form of solution, it would take a huge number of variables (e.g. 10^{50}) to parameterize the class of reasonable storage functions and this would destroy all chance at computation. Thus, we must restrict attention to some very limited class of viscosity subsolutions.

With this in mind, we call a piecewise quadratic function $\varepsilon : R^n \longrightarrow R$ a **Δ-tame quadratic function** provided it is smooth off of the set Δ and continuous on all of R^n. In this article we analyze viscosity subsolutions ε of $H^{*W} \leq 0$ which are Δ-tame quadratic functions. Also, when the origin $v = 0$ is a unique local minimum of ε, we say that ε **dominates** [2] the system (1.1). The function is called strictly positive provided $\varepsilon > 0$ except at $v = 0$ where we have $\varepsilon(0) = 0$. The next theorem adapts [J] [BH] to our setting.

Theorem 1.1. *If $H^{*W} \leq 0$ has a* **positive** *Δ-tame quadratic viscosity subsolution ε, then the system (1.1) is IO-dissipative. If $H^{*W} < 0$, and if ε is strictly positive and dominates (1.1), then the system (1.1) is asymptotically stable. Conversely, if the system (1.1) is IO-dissipative and asymptotically stable, then $H^{*W} \leq 0$ has a non-negative continuous viscosity subsolution ε which has no strict local minima other than $v = 0$.*

Note in the converse statement there is no mention that ε is piecewise quadratic. Even if it is, there is no reason to believe that ε has two quadratic pieces or that all corners are forced to lie on Δ. Possibly no piecewise quadratic subsolution will exist, but a subsolution ε will exist. In this case performance level 1 is achievable in principle, but this will not be determinable from Theorem 1.1. Thus we are performing a conservative analysis of IO dissipativity. It is born of a major pessimism that, in practice, many viscosity

[2] The domination condition is helpful later when we use ε as a Lyapunov function.

subsolutions for $H^{*W} \leq 0$ will be too complicated to be computable. Consequently, we must quickly turn our attention to special classes of simple ε and search thru a class for the best ε. This will produce a controller; but it does not produce optimal performance in an absolute sense.

c. Outline of Results

The paper does several things.

Ultimately it gives a practical numerical scheme for testing if a piecewise linear system is IO -dissipative and stable. The test is conservative in that it amounts to a sufficient but not necessary condition. To be more specific, we give a (necessary and sufficient) test to determine if a given piecewise quadratic HJBI inequality of the form (1.4-6) has an Δ-tame and dominating solution ε. The nature of the test is to solve some linear matrix inequalities (LMI's). See Recipe 4.1 and Theorem 4.2.

In Section 6, we show how a conventional sort of antiwindup device fits into this framework so that the test given in Section 4 applies to determine if it has satisfactory performance. A much harder question is how would one design an antiwindup device so that our test gives a positive answer. We did not make much progress on exact solutions to this problem. Possibly the method here could be used as a starting point for compromises in design.

Also of mathematical interest is the introduction of a very natural type of piecewise Riccati inequality (Section 2c). There is of course an impressive theory of ordinary Riccati equations which is currently being done by linear analysts. What we have here is one of the simplest and most natural nonlinear generalizations. It patches together two "linear pieces" and seems like a reasonable stepping stone from linear to nonlinear theory. Also any progress on it would be very valuable. What we give here is a numerical method for solving these.

We should emphasize that the mathematical difficulty throughout the paper is imposed by the requirement that ε be continuous. Without this the pieces of the problem would be coupled so loosely that getting simultaneous solutions would be easy. Continuity as previously mentioned is a physical requirement.

On a more philosophical note we add that the results in this paper conform to one direction nonlinear H^∞ control will probably take. One must solve nonlinear HJBI *inequalities* not equalities for an unknown function ε. One can pick a low parameter family $\mathcal{G}$ of functions ε as candidates for a solution and find numerical schemes for solving the inequalities. Any time you succeed in finding a nice ε in $\mathcal{G}$ then one has a solution to the control problem. The method is not sharp because we have a low parameter family, but it could easily produce practical controllers. We were lead to believe by Proposition 2.4 that solving HJBI inequalities rather than equalities is critical for this to work. The example shows for the particular family $\mathcal{G}$ we study in this paper that there rarely will be a solution to the HJBI equality. We doubt one will even get close. However, inequality solutions are quiet feasible. The point is that solutions to the HJBI equalities might be much more "complicated functions" than solutions to the corresponding inequalities.

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Section 2. PIECEWISE RICCATI INEQUALITIES

The goal of this section is to reformulate the problem in Section 1 more concretely. This leads us to introduce a class of inequalities which could naturally be called piecewise Riccati inequalities (PRIE's). The objective of Section 4 and 5 is to give a method for solving them.

a. Δ-tame quadratic functions

Piecewise quadratic functions, which vanish at $v = 0$ and are locally minimized there, all have the form

$$\varepsilon = \begin{cases} \varepsilon_1 & \text{if} \quad cv \leq 1 \\ \varepsilon_2 & \text{if} \quad cv > 1 \end{cases} \qquad \text{where} \tag{2.1a}$$

$$\begin{aligned} \varepsilon_1(v) &= v^T E_1 v \\ \varepsilon_2(v) &= v^T E_2 v + Lv - \sigma \end{aligned} \tag{2.1b}$$

Here L is a row vector and σ is a scalar.

Theorem 2.1. *ε is continuous across $\Delta = \{v : cv = 1\}$, that is ε is Δ-tame, if and only if*

$$E_2 = E_1 + \frac{L^T c + c^T L}{2} + \sigma c c^T. \tag{2.2}$$

Proof: Necessity: Since ε is continuous, when $cv = 1$ we must have

$$\varepsilon_2 - \varepsilon_1 = v^T (E_2 - E_1) v + Lv - \sigma = 0, \quad \forall v \in \Delta. \tag{2.3}$$

With the parametrization $v = \mu + \kappa$ for $cv = 1$, where $c\mu = 0$ and $c\kappa = 1$, it follows from (2.3) that

$$\mu^T (E_2 - E_1)\mu + 2\mu^T (E_2 - E_1)\kappa + \mu^T L^T + \kappa^T L^T + \kappa^T (E_2 - E_1)\kappa - \sigma = 0 \tag{2.4}$$

Equation (2.4) holds for all μ satisfying $c\mu = 0$. This implies that

$$\mu^T (E_2 - E_1)\mu = 0, \tag{2.5}$$

$$2\mu^T(E_2 - E_1)\kappa + \mu^T L^T = 0. \tag{2.6}$$

Applying Lemma A.1 to (2.5) we have

$$E_2 - E_1 = c^T M^T + Mc, \tag{2.7}$$

where M is an arbitrary matrix with appropriate dimension. Plugging (2.7) into (2.6), we get

$$L^T = -2M + \sigma_1 c^T, \tag{2.8}$$

where σ_1 is a scalar. Thus (2.4) is reduced to

$$\kappa^T(E_2 - E_1)\kappa + L\kappa - \sigma = 0.$$

Plugging (2.7) (2.8) into the above and recalling that $c\kappa = 1$, we get $\sigma = \sigma_1$. Solve (2.8) for M in terms of L, σ and substitute this into (2.7) to get (2.2).

Sufficiency can be easily shown by verifying the continuity of ε across Δ.

In summary, we have ε is Δ-tame if and only if it has the following form:

$$\varepsilon(v) = v^T E_1 v \quad if \;\; cv < 1,$$

$$\varepsilon(v) = v^T(E_1 + \frac{L^T c + c^T L}{2} + \sigma c c^T)v + Lv - \sigma \quad if\; cv > 1,$$

where L is an arbitrary row vector and σ is an arbitrary scalar. ∎

b. Subsolutions

We seek Δ-tame quadratic ε satisfying

$$H^{*W}(v,p) \le 0 \tag{2.9}$$

for all $v \in R^n$ and $p \in \nabla^+\varepsilon(v)$. Consequently, the first step is to make (2.9) more concrete.

Lemma 2.2. *The Δ-tame quadratic function ε satisfies (2.8) if and only if*

$$\begin{aligned} H^{*W}(v, \nabla\varepsilon_1(v)) &\le 0 \quad \text{if} \quad cv \le 1, \\ H^{*}(v, \nabla\varepsilon_2(v)) &\le 0 \quad \text{if} \quad cv \ge 1. \end{aligned} \tag{2.10}$$

Moreover, ε is strictly positive and dominating if and only if $E_1 > 0$, $E_2 > 0$ and $cE_2^{-1}L^T \ge -2$

The proof uses the following straightforward characterization of the superdifferential $\nabla^+\varepsilon(v)$.

Lemma 2.3.

$$\nabla^{+}\varepsilon(v)=\begin{cases}\nabla\varepsilon_1(v) & \text{if } cv<1\\ \text{convex hull of } \nabla\varepsilon_1(v) \text{ and } \nabla\varepsilon_2(v) \text{ or empty} & \text{if } cv=1\\ \nabla\varepsilon_2(v) & \text{if } cv>1\end{cases} \tag{2.11}$$

Proof of Lemma 2.2: The necessity of (2.10) is trivial. To show the sufficiency of (2.10), notice that the function ε satisfies (2.9) provided $\nabla\varepsilon$ satisfies
$H^{*W}(v,\nabla\varepsilon(v))\leq 0$ when $cv\neq 1$ and when $cv=1$ it satisfies no constraint or

$$H^{*W}(v,\alpha\nabla\varepsilon_1(v)+(1-\alpha)\nabla\varepsilon_2(v))\leq 0 \tag{2.12}$$

for all $0\leq\alpha\leq 1$. Now, H^{*W} is convex in p, so

$$H^{*W}(v,\alpha\nabla\varepsilon_1(v)+(1-\alpha)\nabla\varepsilon_2(v))\leq \alpha H^{*W}(v,\nabla\varepsilon_1(v))+(1-\alpha)H^{*W}(v,\nabla\varepsilon_2(v))\leq 0$$

Thus (2.12) holding for all α is equivalent to (2.10).

Now we analyze when ε is dominating. Since ε is continuous, it is strictly positive and dominating if and only if $E_1>0$, $E_2>0$ and ε_2 has its local minimum on $cv\leq 1$ rather than in $cv>1$. That is, the minimum value of the function $\varepsilon_2(v)=v^TE_2v+Lv-\sigma$ is achieved at $v=v_{min}$ such that $cv_{min}\leq 1$. The minimizer v_{min} of the function $\varepsilon_2(v)$, satisfies

$$2E_2v_{min}+L^T=0.$$

Since $E_2>0$ is invertible, we get

$$v_{min}=-\frac{1}{2}E_2^{-1}L^T.$$

Thus we have

$$-\frac{1}{2}cE_2^{-1}L^T\leq 1,$$

i.e., $cE_2^{-1}L^T\geq -2$, and we have proved Lemma 2.2. ∎

c. Piecewise Riccati inequalities

This section summarizes what has gone before in that it lays out exactly the class of equations we want to solve and to study.

We have given matrices A,B,C,a,b,c. Define

$$\mathcal{R}_1(E_1)=E_1A+A^TE_1+E_1BB^TE_1+C^TC,$$

$$\mathcal{R}_2(E_2) = (A-ac)^T E_2 + E_2(A-ac) + (C-bc)^T(C-bc) + E_2 BB^T E_2$$

The unknowns E_1, L, σ are said to satisfy a **piecewise Riccati inequality** provided:

(i) E_1 satisfies a conventional Riccati inequality

$$v^T \mathcal{R}_1(E_1) v \leq 0 \ \forall v \tag{2.13}$$

together with an affine quadratic inequality

$$v^T \mathcal{R}_2(E_2) v + v^T \mathcal{L}_2 + \mathcal{L}_2^T v + \mathcal{K} \leq 0, \tag{2.14}$$

which must hold for all v in the halfplane $cv \geq 1$. This imposes a restriction on E_1, L and σ.

Here

$$\begin{aligned} E_2 &= E_1 + \frac{L^T c + c^T L}{2} + \sigma c c^T, \\ \mathcal{L}_2 &= (A-ac)^T \frac{L^T}{2} + (C-bc)^T b + E_2^T (a + BB^T \frac{L}{2}), \\ \mathcal{K} &= \frac{La + a^T L}{2} + b^T b + \frac{LB^T BL^T}{4}. \end{aligned} \tag{2.15}$$

An **equivalent statement** which is more matrix theoretic simply replaces (2.14) by

$$[v^T \ 1] \ \Pi_2 \begin{bmatrix} v \\ 1 \end{bmatrix} \leq 0 \tag{2.14$'$}$$

for all $cv \geq 1$, where

$$\Pi_2 = \begin{bmatrix} \mathcal{R}_2(E_2) & \mathcal{L}_2 \\ \mathcal{L}_2^T & \mathcal{K} \end{bmatrix}$$

The objective of the next part of the paper is given a piecewise Riccati inequality find E_1,L and σ with $E_1 > 0$, $E_2 > 0$ solving it. This is motived by the the derivation in Sections 2a, 2b which says that $H^{*W}(v, \nabla\varepsilon(v)) \leq 0$ is equivalent to (2.13) for $cv \leq 1$ and (2.14) for $cv \geq 1$. Thus the key thing we must solve is a piecewise Riccati inequality. Also, we shall be interested in imposing the side condition that ε is dominating.

d. Equalities vs inequalities

In this subsection we point out that Δ-tame solutions to $H^{*W} = 0$ rather than to $H^{*W} \leq 0$ are rare. One of the most useful properties of linear Riccati inequalities is that if a Riccati inequality has a positive definite solution E, then the Riccati equality will also have a stabilizing positive semi-definite solution $\tilde{E}$, c.f.

Theorem 2.2 **[PAJ]**. This allows one to compute the stabilizing solution to the Riccati equality and never worry that we might have achieved better performance by solving a Riccati inequality. Indeed commercial computer programs only solve Riccati equations and not inequalities. These are precisely what is needed for the H^∞ control problem.

In this paper, in order to have a chance at actually computing solutions we are only considering Δ-tame viscosity subsolutions and the piecewise Riccati inequalities corresponding to them. The next lemma shows that piecewise Riccati inequalities (2.13) (2.14) rarely have Δ-tame viscosity subsolutions with equality.

Proposition 2.4. *If $cB \neq 0$ and for all $v \in \Delta$*

$$H_1^{*W}(v, \nabla\varepsilon_1(v)) = H_2^{*W}(v, \nabla\varepsilon_2(v)) = 0,$$

then $E_2 = E_1 + \gamma_1 c^T c$ for some real number γ_1, and $\nabla\varepsilon_2(v) = 2E_2 v + \gamma_2 c^T$ for some real number γ_2.

Proof: See Section 5.5. ∎

Using the parameterization $v = \mu + \kappa$, where $c\mu = 0$ and $c\kappa = 1$, we have the following formula

$$H_2^{*W}(v, \nabla\varepsilon_2(v)) = (\mu+\kappa)^T \mathcal{R}_2(E_2)(\mu+\kappa) + 2\mu^T \mathcal{L}_2 + 2\kappa^T \mathcal{L}_2 + \mathcal{K} = 0 \tag{2.16}$$

for all μ. Since $\mu^T \mathcal{R}_2 \mu = 0$, and from $H_2^{*W}(v, \nabla\varepsilon_2(v)) \leq 0$ for all v satisfying $cv > 1$, we know that $\mathcal{R}_2(E_2) \leq 0$. It follows that $\mathcal{R}_2(E_2)\mu = 0$. Thus, (2.16) becomes

$$\kappa^T \mathcal{R}_2(E_2)\kappa + 2\mu^T \mathcal{L}_2 + 2\kappa^T \mathcal{L}_2 + \mathcal{K} = 0.$$

We deduce from the above equation that

$$\mathcal{L}_2 = \gamma_2 c^T$$

where γ_2 is a scalar. Therefore, E_1, L, γ_1 and γ_2 must satisfy the following equations

$$\mathcal{R}_1(E_1) = 0, \quad P\mathcal{R}_2(E_1 + \gamma_1 c^T c)P = 0,$$

$$(A - ac)^T \frac{L^T}{2} + (C - bc)^T b + (E_1 + \gamma_1 c^T c)(a + BB^T \frac{L}{2}) = \gamma_2 c^T.$$

In general, these equations are over determined.

Suppose $H_1^{*W} = H_2^{*W}$ for all v, then $\mathcal{L} = 0$ and $\mathcal{K} = 0$ as well. Notice that $cB \neq 0$ and $\mathcal{K} = 0$ depends only on the system parameters. Clearly, for most systems, we have $cB \neq 0$ and $\mathcal{K} \neq 0$, in which case, the above proposition says that the piecewise HJI equalities $H_1^{*W} = 0$, $H_2^{*W} = 0$ have no Δ-tame solutions.

Even if $\mathcal{K} = 0$, as long as $cB \neq 0$, the constraints on E_1 are in general over determined since E_1, L, γ_1, must satisfy

$$\mathcal{R}_1(E_1) = 0, \quad \mathcal{R}_2(E_1 + \gamma_1 c^T c) = 0,$$

$$(A - ac)^T \frac{L^T}{2} + (C - bc)^T b + (E_1 + \gamma_1 c^T c)(a + BB^T \frac{L}{2}) = 0.$$

Therefore, we conclude that solutions to the piecewise Riccati equations rarely exist.

Section 3. SOLVING PIECEWISE RICCATI INEQUALITIES

Our goal is to express a piecewise Riccati inequality (2.13-15) in terms of *matrix inequalities* constraining the unknown variables E_1, E_2, L and σ. We express these conditions as matrix inequalities because these are susceptible to optimization algorithms using barrier functions **[BGFB]**. Our derivation first gives matrix inequalities in which the unknowns enter quadratically (QMI), then we convert these to linear matrix inequalities (LMI). Before stating a Recipe we describe the rather simple skeleton underlying it's proof.

The Riccati type expression H_1^{*W} of (2.13) converts to an LMI by standard techniques, see **[BGFB]** or later parts of this paper.

The less usual thing is

$$H_2^{*W}(v, \nabla \varepsilon_2(v)) \tag{3.1}$$

of (2.14). The main point is that $H_2^{*W}(v, \nabla \varepsilon_2(v))$ is an affine quadratic form and we must determine negativity in the half space $cv \geq 1$. This is not the usual positive definite matrix condition.

Fortunately, a very clever trick known as the S-procedure (c.f. p23 **[BGFB]**) converts this to a QMI which we later convert to an LMI. This S-procedure in our context is summarized by

Lemma 3.1. *Let f denote an affine quadratic function*

$$f(v) = v^T \mathcal{R} v + \mathcal{L}^T v + \psi,$$

where v is a vector in R^n, $\mathcal{R}$ is a symmetric matrix, $\mathcal{L}$ and ψ are matrices of appropriate dimensions.

(i) $f(v) \leq 0$ for all v is equivalent to $\Pi_f \leq 0$ where

$$\Pi_f = \begin{bmatrix} \mathcal{R} & \frac{1}{2}\mathcal{L} \\ \frac{1}{2}\mathcal{L}^T & \psi \end{bmatrix}$$

(ii) $f(v) \leq 0$ *for all* $cv \geq 1$ *if and only if there exists a number* $\tau \geq 0$ *so that*

$$\Pi_f + \tau \begin{bmatrix} 0 & \frac{c^T}{2} \\ \frac{c}{2} & -1 \end{bmatrix} \leq 0$$

When we apply this to the piecewise Riccati inequalities (2.13) (2.14) and (2.15) we immediately get.

RECIPE 3.2 QMI. *To find a* Δ*-tame quadratic function which is a viscosity subsolution to* $H^{*W} \leq 0$ *one can solve the following QMI's for* E_1, L, σ

(i) $\mathcal{R}_1(E_1) \leq 0,$

(ii) There exists $\tau \geq 0$ *so that*

$$\Pi_2 + \tau \begin{bmatrix} 0 & \frac{c^T}{2} \\ \frac{c}{2} & -1 \end{bmatrix} \leq 0.$$

That is

$$\begin{bmatrix} \mathcal{R}_2(E_2) & \mathcal{L}_2 \\ \mathcal{L}_2^T & \mathcal{K} \end{bmatrix} + \tau \begin{bmatrix} 0 & \frac{c^T}{2} \\ \frac{c}{2} & -1 \end{bmatrix} \leq 0, \quad \text{where} E_2 = E_1 + \frac{L^T c + c^T L}{2} + \sigma c c^T,$$

and $\mathcal{R}_2, \mathcal{L}_2, \mathcal{K}$ *are defined in Section 2c.*

(iii) $E_1 \geq p_1 I, \quad E_2 \geq p_2 I,$

where p_1, p_2 *are preassigned positive numbers insuring the positivity of* E_1, E_2.

Moreover, ε *is strictly positive and dominating provided* $cE_2^{-1}L^T \geq -2$. *(This is not a QMI and compromise solutions will be discussed later.)*

Section 4. SPECIFIC FORMULAS FOR SUBSOLUTIONS; LMI

This section contains a detailed method for solving piecewise Riccati inequalities. This is the main result of the paper. Recall that the unknowns are E_1, L and σ. Of course, E_2 is constructed from these by

$$E_2 = E_1 + \frac{L^T c + c^T L}{2} + \sigma c c^T.$$

The crucial matrix is

$$\Omega := \begin{bmatrix} (A - ac)^T E_2 + E_2(A - ac) + (C - bc)^T (C - bc) & \mathcal{L}_2 & E_2 B \\ \mathcal{L}_2^T & \mathcal{K} & 0 \\ B^T E_2 & 0 & -I \end{bmatrix}$$

RECIPE 4.1. *To find a Δ-tame quadratic function, which is a viscosity subsolution to $H^{*W} \leq 0$, one can solve the following LMI's for E_1, L, σ and τ.*

(i)

$$\begin{bmatrix} E_1 A + A^T E_1 + C^T C & E_1 B \\ B^T E_1 & -I \end{bmatrix} \leq 0,$$

(ii)

$$\Omega + \tau \begin{bmatrix} 0 & c^T/2 & 0 \\ c/2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leq 0,$$

(iii)

$$E_1 \geq p_1 \ I, \quad E_2 \geq p_2 \ I,$$

where p_1, p_2 are preassigned positive numbers.

Moreover, ε is strictly positive and dominating provided $cE_2^{-1}L^T \geq -2$. (This is not an LMI.)

Theorem 4.2. *If a Δ-tame quadratic function satisfying (iii) which is a viscosity subsolution to $H^{*W} \leq 0$ exists, then Recipe 4.1 finds it.*

There are various compromises one might try to enforce the condition that ε is dominating. One very crude simple compromise is to impose

$$-\sigma = \varepsilon_2(0) \leq \min_{v \in \Delta} \varepsilon_2(v),$$

which is easily convertible to an LMI. Now 0 is an arbitrarily chosen point in $\{v : cv \leq 1\}$. One could improve this location at subsequent iterations if need be.

Section 5. PROOFS

This section gives proofs for Theorem 4.2 and Proposition 2.4. Theorem 4.2 says that Recipe 4.1 solves (2.12) and (2.13) with a solution of prescribed positivity.

a. Proof of Theorem 4.2

All that we must do is reduce Recipe 4.1 to Recipe 3.2. We use the next lemma repeatedly.

Lemma 5.1. *Let F_1 be a symmetric $n \times n$ matrix and F_2 be an $n \times n$ matrix. The following two conditions are equivalent*

(1) $F_2^T F_2 + F_1 \leq 0$;

(2)

$$\begin{bmatrix} F_1 & F_2^T \\ F_2 & -I \end{bmatrix} \leq 0$$

Proof: Define the following matrix

$$T = \begin{bmatrix} I & 0 \\ F_2 & I \end{bmatrix}$$

We have

$$T^T \begin{bmatrix} F_1 & F_2^T \\ F_2 & -I \end{bmatrix} T = \begin{bmatrix} F_2^T F_2 + F_1 & 0 \\ 0 & -I \end{bmatrix}$$

Since T is nonsingular, we conclude that (1) is equivalent to (2). ■

Equivalence of item (i) in the Recipes 3.2 and 4.1 follows directly from Lemma 5.1. To prove equivalence for item (ii), write Π_2 as its pure quadratic part plus affine linear part, then factor its quadratic part as $\hat{Q}\hat{Q}^T$ to obtain

$$\Pi_2 = \hat{L} + \hat{Q}\hat{Q}^T$$

where

$$\hat{Q} = \begin{bmatrix} E_2 B \\ 0 \end{bmatrix} \qquad \hat{L} = \begin{bmatrix} (A - ac)^T E_2 + E_2(A - ac) + (C - bc)^T (C - bc) & \mathcal{L}_2 \\ \mathcal{L}_2^T & \mathcal{K} \end{bmatrix}$$

Thus the matrix in Recipe 3.2 (ii) is

$$\hat{L} + \hat{Q}\hat{Q}^T + \tau \begin{bmatrix} 0 & c^T/2 \\ c/2 & -1 \end{bmatrix} \leq 0. \tag{5.1}$$

Observe that Ω in Recipe 4.1 is $\begin{bmatrix} \hat{L} & \hat{Q} \\ \hat{Q}^T & -I \end{bmatrix}$. Therefore Lemma 5.1 says that the inequality (5.1) is equivalent to Recipe 4.1 (ii). ■

b. Proof of Proposition 2.4

In this subsection, we use Theorem 2.1 to prove Proposition 2.4 which asserts that equality solutions to piecewise Riccati equalities are rare.

Let P denote the orthogonal projection onto the null space of c. That is, $cP = 0$.

Write $E_2 = E_1 + \phi$ where $\phi = Mc + c^T M^T$, as was done in (2.7). The quadratic terms of H_1^{*W} and H_2^{*W} are $\mathcal{R}_1(E_1)$ and $\mathcal{R}_2(E_2)$. Note that

$$P\mathcal{R}_2(E)P = P\mathcal{R}_1(E)P \leq 0$$

It follows from the assumption that

$$P\mathcal{R}_1(E_1)P = 0, \;\; P\mathcal{R}_2(E_2)P = 0.$$

Now, write $\mathcal{R}_2(E_2)$ as

$$\mathcal{R}_2(E_2) \quad = \mathcal{R}_2(E_1) + \phi(A - ac + BB^T E_1) + (A - ac + BB^T E_1)^T \phi + \phi BB^T \phi$$

so, multiply by P on both sides to get

$$0 = PR_2(E_1)P + P\phi(A - ac + BB^T E_1)P + P(A - ac + BB^T E_1)^T \phi P + P\phi BB^T \phi P.$$

Since $cP = 0$ and $PR_2(E_1)P = PR_1(E_1)P = 0$ we have

$$P\phi(A + BB^T E_1)P + P(A + BB^T E_1)^T \phi P + P\phi BB^T \phi P = 0.$$

Plugging $\phi = Mc + c^T M^T$, and using $cP = 0$, we get

$$PMc(A + BB^T E_1)P + P(A + BB^T E_1)^T c^T M^T P + PMcBB^T c^T M^T P = 0,$$

or

$$PMg + g^T M^T P + PMGM^T P = \quad 0$$

where $g = c(A + BB^T E_1)P$ and $G = cBB^T c^T$. Complete the square to get

$$(G^{-1}g + [PM]^T)^T G(G^{-1}g + [PM]^T) - g^T G^{-1} g = 0,$$

which implies that $G^{-1}g + [PM]^T = G^{-1}g$. This proves that $PM = 0$, which says that M is a scalar multiple of c^T. The conclusion of this lemma follows from (2.7), (2.8) and (2.1b). ∎

Section 6. SYSTEMS WITH SATURATION

Analysis of the performance of many linear systems with saturation nonlinearities leads directly to the problem described in (1.1). We illustrate this with one example.

a. Performance Measures

For linear H^∞ control, performance measures often take the mathematical form of mixed sensitivity. This can be expressed in terms of systems filtering measurements on the closed-loop system as

$$\text{mixed sensitivity} = \left\| \begin{pmatrix} SYS_1(output - input) \\ SYS_2(output) \end{pmatrix} \right\|^2$$

Such performance measures with linear systems SYS_1 and SYS_2 are the traditional mixed sensitivity objectives of H^∞ control. If a plant is piecewise linear, then it is natural to take SYS_1, and SYS_2 to be piecewise linear with the same corner locations as the plant. This would be true for rather general configurations, such as the antiwindup structures we address soon.

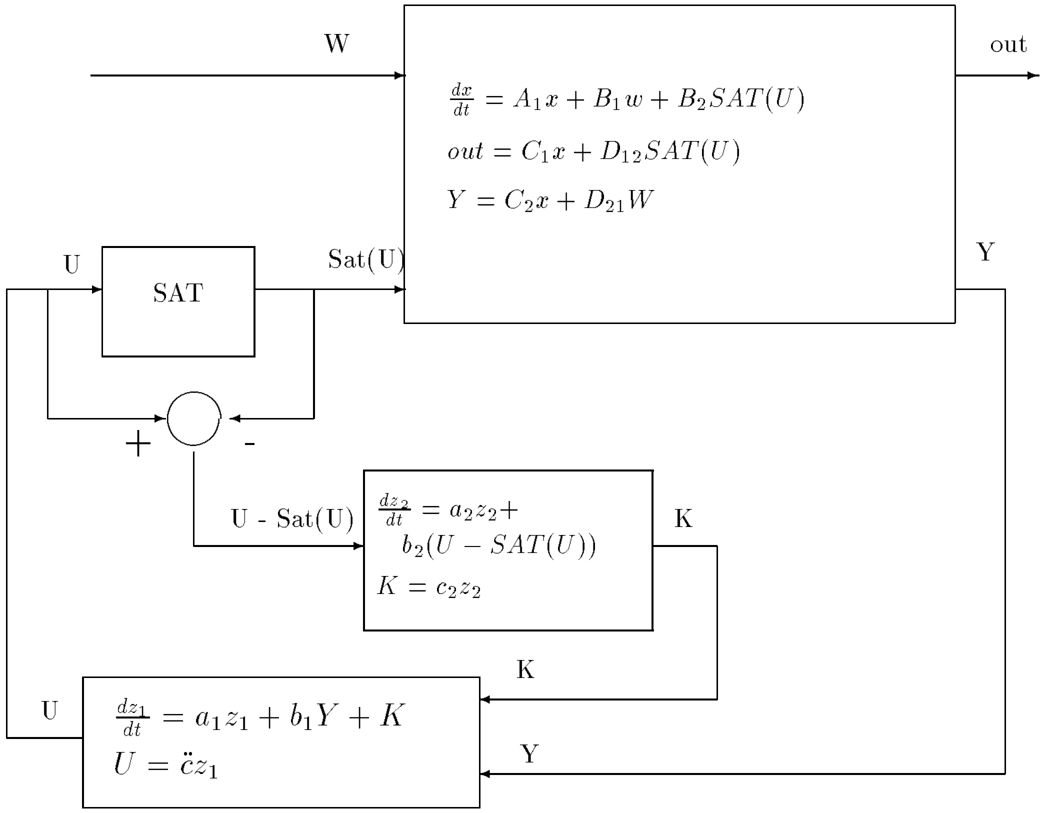
Our main point here is that any of these situations lead directly to math problems of the type in (1.1). Now, we turn to a more specific situation.

b. Antiwindup devices

The article of Campo and Morari [CM] describes an architecture which embraces very common control systems for systems with saturation. These are called antiwindup devices. While we have not been careful in laying out the performance specs for the problem, we think it illustrates how saturation problems with performance fit into the mathematical framework of this article. Even if one takes greater care with the physics of the problem, we expect that this type of math problem will lie at its core.

Let Sat denote the classical saturation function

$$Sat(U) = \begin{cases} U & \text{if } -1 < cv < 1 \\ 1 & \text{if } cv \geq 1 \\ -1 & \text{if } cv \leq -1 \end{cases}$$



The closed-loop system looks like this:

$$
\begin{aligned}
dx/dt &= Ax + B_1 W + B_2 Sat(U) \\
dz_1/dt &= a_1 z_1 + b_1 Y + c_2 z_2 \\
dz_2/dt &= a_2 z_2 + b_2[U - Sat(U)] \\
Y &= C_2 x + D_{21} W \\
out &= C_1 x + D_{12} Sat(U) \\
U &= \bar{c} z_1
\end{aligned}
$$

where A_0, B_1, B_2, C_1, C_2, D_{21} and D_{12} are given (determined by the plant and performance weightings) and a_1, b_1 are designed based on the linear system without consideration of the saturation nonlinearity. The state of the closed-loop system is $v = (x, z_1, z_2)$. The antiwindup device corresponds to a_2, b_2 and c_2.

We can easily write the closed-loop system into a form of (1.1). The following are the coefficients of system (1.1):

$$\tilde{A}_1 = \begin{bmatrix} A & B_2\bar{c} & 0 \\ b_1C_2 & a_1 & c_2 \\ 0 & 0 & a_2 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} B_1 \\ b_1D_{21} \\ 0 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} A & 0 & 0 \\ b_1C_2 & a_1 & c_2 \\ 0 & b_2\bar{c} & a_2 \end{bmatrix}, \quad \tilde{a}_2 = \begin{bmatrix} B_2 \\ 0 \\ -b_2 \end{bmatrix},$$

$$\tilde{C}_1 = [C_1 \;\; D_{12}\bar{c} \;\; 0], \quad c = [0 \;\; \bar{c} \;\; 0], \quad \tilde{B}_2 = \tilde{B}_1, \;\; \tilde{C}_2 = [C_1 \;\; 0 \;\; 0], \;\; \tilde{b}_2 = D_{12}.$$

It can be verified that $\tilde{A}_2 = \tilde{A}_1 - \tilde{a}_2 c$ and $\tilde{C}_2 = \tilde{C}_1 - \tilde{b}_2 c$. Thus, in system (1.1), we have $a = \tilde{a}_2$ and $b = \tilde{b}_2$. Also, $\tilde{D}_1 = \tilde{D}_2 = 0$.

A discrepancy between these systems and those of (1.1) is that these systems have two discontinuity sets $cv = -1$ and $cv = 1$ instead of one. This does not seem to impose a difficulty since one could use the mathematics at hand on corners in a different location.

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APPENDIX 1

Lemma A.1. *Given a matrix B with full row rank and a symmetric matrix A. Suppose for all y satisfying $By = 0$ we have $y^T Ay = 0$. Then there exists a matrix M such that*

$$A = B^T M^T + MB.$$

Proof. Since B has full row rank, there exists a nonsingular matrix S such that $BS = [I \quad 0]$.

Denote

$$S^T AS = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}.$$

Since for all x satisfying $BSx = 0$ we have $x^T S^T ASx = 0$, it follows that $A_{22} = 0$. Clearly, we have

$$S^T AS = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}A_{11} \\ A_{12}^T \end{bmatrix} [\, I \quad 0 \,] + \begin{bmatrix} I \\ 0 \end{bmatrix} [\, \tfrac{1}{2}A_{11} \quad A_{12} \,]$$

Let

$$N = \begin{bmatrix} \frac{1}{2}A_{11} \\ A_{12}^T \end{bmatrix} \qquad M = (S^{-1})^T N.$$

We have $S^T AS = NBS + S^T B^T N^T = S^T MBS + S^T B^T M^T S$. Since S is nonsingular, the above equation is equivalent to

$$A = B^T M^T + MB.$$

■

APPENDIX 2: ANOTHER WAY OF SOLVING PRIE's

In the first version of this manuscript we had a different way of solving Riccati inequalities. It was messier to write down but was sufficiently different that the method is worth mentioning.

Possibly the most novel outcome of this approach was the following idea which we sketch very roughly. This is an alternative to the conditions in RECIPE 4.1 which would be solved differently than would LMI's. Possibly a combination of this and optimization of small size MI's would be rewarding. Also we mention that the approach given in this appendix involves one less unknown (τ), thus from the computational point of view it maybe easier to compute.

It is easy to show that for the piecewise Riccati inequalities in Section 2c to hold two ordinary Riccati inequalities $\mathcal{R}_1(E_1) \leq 0, \mathcal{R}_2(E_2) \leq 0$ must hold. There are additional constraints which we do not elaborate in this paper but which can be found in an an earlier version of this paper obtainable via ftp.

a. Another Approach to Solving Riccati Inequalities

Recall [W] that Riccati equations for dissipative systems have two distinguished solutions X_a, and X_r, which have the natural physical interpretation in terms of the associated available and required energy storage functions $\varepsilon_a(v) = v^T X_a v$ and $\varepsilon_r(v) = v^T X_r v$:

$\varepsilon_a(v) =$ maximum energy available from the system when started at state v

$\varepsilon_r(v) =$ minimum energy required to drive the system from state 0 to state v.

A standard fact is: for a dissipative reachable system, $\varepsilon_a(v) \leq \varepsilon(v) \leq \varepsilon_r(v)$ holds for all v and for any nonnegative storage function ε. In terms of matrices, this becomes $0 \leq X_a \leq X \leq X_r$ in the sense of matrix inequalities.

This can be used to analyze the conditions $\mathcal{R}_1(E_1) \leq 0$ and $\mathcal{R}_2(E_2) \leq 0$, which are an important part of Recipe 3.2, as follows:

Step 1. Find the available and required storage solutions

$X_{a_1}, X_{r_1}, X_{a_2}, X_{r_2}$ to $\mathcal{R}_j(X_{a_j}) = 0 = \mathcal{R}_j(X_{r_j})$.

Step 2. Find a positive matrix E_0 simultaneously satisfying the linear matrix inequalities

$$PX_{a_1}P \leq E_0 \leq PX_{r_1}P \quad \textit{and}$$

$$PX_{a_2}P \leq E_0 \leq PX_{r_2}P \tag{A2.1}$$

We wish to build $E_1 = E_0 + \phi_1$ and $E_2 = E_0 + \phi_2$, where $\phi_j = M_j c + c^T M_j^T$ for some vectors $M_j \in R^n$. In order for (i) (iia) to hold, M_j must satisfy

$$X_{a_1} - E_0 \leq \phi_1 \leq X_{r_1} - E_0 \tag{A2.a}$$

$$X_{a_2} - E_0 \leq \phi_2 \leq X_{r_2} - E_0 \tag{A2.b}$$

These are particularly simple linear matrix inequalities since $v^T \phi_j v = (cv^T)(M_j v)$. Thus, we finish with

Step 3. To solve $\mathcal{R}_1(E_1) \leq 0$ and $\mathcal{R}_2(E_2) \leq 0$ solve (A2.1).

As already mentioned simultaneously solving the two Riccati inequalities $\mathcal{R}_1 \leq 0$ and $\mathcal{R}_2 \leq 0$ is not enough. The remaining conditions do not effect M_1, that is, the choice of M_1, but impose nonlinear constraints on M_2 and σ which after considerable development can be treated as LMI's.

b. A Lemma

A full theory extending what we just described can be based on replacing part (ii) of Lemma 3.1 with a test sketched in this subsection. We do not give a derivation of the test because it involves messy specialized formulas. Also the lemma here is possibly of independent use in that a variety of LMI solutions of piecewise Riccati inequalities can be based on it.

This method places an emphasis on the set Δ along which our corner occurs. With this in mind we write H^{*W} as a quadratic function in distinguished variables $t \in R, \mu \in R^n$ with $c\mu = 0$. Set

$$f_2(t,\mu) := H_2^{*W}([t+1]\kappa + \mu, \nabla\varepsilon_2([t+1]\kappa + \mu)) \qquad (A2.3)$$

where $c\kappa = 1$ and $\mu^T\kappa = 0$. Here you see that we have made the replacement $v = [t+1]\kappa + \mu$ which expresses v in coordinates [3] transversal to Δ and along Δ (actually the tangent space to Δ). Property (2.13) of H_2^{*W} translates to

$$f_2(\mu,t) \leq 0 \quad for \quad t \geq 0.$$

The next lemma converts this type of condition to two sets of QMI's. These are straightforward to translate to LMI's.

Lemma A2.1 If the vectors v on which f acts are partitioned $(\tilde{\mu}, t)$ with $\tilde{\mu} \in R^{n-1}$ and $t \in R$, and f is expressed as

$$f(\tilde{\mu},t) := [\tilde{\mu}^T,\ t]\begin{bmatrix} \beta & \rho \\ \rho^T & \delta \end{bmatrix}\begin{bmatrix} \tilde{\mu} \\ t \end{bmatrix} + [w_1,\ w_2]\begin{bmatrix} \tilde{\mu} \\ t \end{bmatrix} + \psi$$

we have that Π_f in this partitioning becomes the block matrix

$$\Pi_f = \begin{bmatrix} \beta & \rho & \frac{1}{2}w_1^T \\ \rho^T & \delta & \frac{1}{2}w_2 \\ \frac{1}{2}w_1 & \frac{1}{2}w_2 & \psi \end{bmatrix}$$

Then, $f(\tilde{\mu},t) \leq 0$ for all $t \geq 0$ if and only if one of the following conditions hold

(a) $\Pi_f \leq 0$;

[3] A byproduct of this coordinatization which might play a role in antiwindup *design* is

$$f_2(\mu,0) = H_2^{*W}(v, \nabla\varepsilon_2(v)) = H_1^{*W}(v, \nabla\varepsilon_2(v)) \quad \forall v \in \Delta$$

and that H_1^{*W} does not depend on a, b! Thus $f_2(\mu,0) \leq 0$ is a (very strong) set of constraints on E_2 which does not depend on a, b.

(b) $\mathcal{R} = 0,\ w_1 = 0,\ w_2 \leq 0,\ \psi \leq 0$;

(c) $\mathcal{P} = 0,\ \rho = 0,\ w_2 \leq 0,\ \delta \leq 0$;

(d) $\mathcal{P} \leq 0$ (but $\neq 0$), $\mathcal{R} \leq 0$ (but $\neq 0$), $w_2 - w_1 \beta^{\#} \rho \leq 0$; where

$$\mathcal{R} = \begin{bmatrix} \beta & \rho \\ \rho^T & \delta \end{bmatrix} \qquad \mathcal{P} = \begin{bmatrix} \beta & \frac{1}{2} w_1^T \\ \frac{1}{2} w_1 & \psi \end{bmatrix} \tag{A2.4}$$

Here $\beta^{\#}$ denotes the pseudoinverse of β.

Proof: See Appendix 3 of the long (unpublished version).

APPENDIX 3: PROOF OF LEMMA A2.1 NOT TO BE PUBLISHED

Proof. (i) Note that the function $f(v)$ can written as

$$f(v) = [v^T \ 1]\Pi_f[v^T \ 1]^T.$$

Thus $\Pi_f \leq 0$ implies $f(v) \leq 0$, $\forall v$. Suppose $f(v) \leq 0$, $\forall v$. Then it is necessary that $\mathcal{R}$ is negative semi-definite. To show $\Pi_f \leq 0$ we need to check that $z^T\Pi_f z \leq 0, \quad \forall z$. Let us discuss two cases, one is $z = (v^T, 0)^T$, and the other is $z = (v^T, \eta)^T$ with $\eta \neq 0$. In case 1, $z^T\Pi_f z = v^T\mathcal{R}v \leq 0$. In case 2, we have

$$z^T\Pi_f z = \eta^2 \begin{bmatrix} \frac{v^T}{\eta} & 1 \end{bmatrix} \Pi_f \begin{bmatrix} \frac{v^T}{\eta} & 1 \end{bmatrix}^T = \eta^2 f(\frac{v}{\eta}) \leq 0.$$

(ii) Sufficiency.

- If (a) holds, then

$$f(\tilde{\mu}, t) = [\tilde{\mu}^T \ t \ 1]\Pi_f[\tilde{\mu}^T \ t \ 1]^T \leq 0, \quad \forall t.$$

- If (b) holds, then

$$f(\tilde{\mu}, t) = w_2 t + \psi \leq 0, \quad \forall t \geq 0.$$

- If (c) holds, then

$$f(\tilde{\mu}, t) = \delta t^2 + w_2 t \leq 0, \quad \forall t \geq 0.$$

- If (d) holds, then $\mathcal{R} \leq 0$ implies that $\delta \leq 0$. We have two cases to discuss.

  Case 1: $\delta = 0$. In this case, we have $\rho = 0$ and $w_2 \leq 0$. Thus,

$$f(\tilde{\mu}, t) = \tilde{\mu}^T\beta\tilde{\mu} + w_1\tilde{\mu} + w_2 t + \psi.$$

  Since $f(\tilde{\mu}, 0) = [\tilde{\mu}^T \ 1]\mathcal{P}[\tilde{\mu}^T \ 1]^T \leq 0$, and $\frac{\partial f}{\partial t} = w_2 \leq 0$, we conclude that $f(\tilde{\mu}, t) \leq 0, \quad \forall t \geq 0$.

  Case 2: $\delta < 0$. Clearly, we have

$$\lim_{t \to +\infty} f(\tilde{\mu}, t) = -\infty.$$

  Also, $f(\tilde{\mu}, 0) = [\tilde{\mu}^T \ 1]\mathcal{P}[\tilde{\mu}^T \ 1]^T \leq 0$. Now, suppose $f(\tilde{\mu}, t_0) > 0$ for some $t_0 > 0$. Then, the following equations must have solution $t^* > 0$ and μ^*

$$0 = \frac{\partial f}{\partial \tilde{\mu}} = 2\beta\tilde{\mu} + 2\rho t + w_1^T$$

$$0 = \frac{\partial f}{\partial t} = 2\delta t + 2\tilde{\mu}^T\rho + w_2$$

The nullspace of β is orthogonal to ρ and w_1; orthogonal to w_1 in order for $\mathcal{P} \leq 0$ to hold and to ρ in order for $\mathcal{R} \leq 0$ to hold. Thus we can get a solution to the first equation

$$\tilde{\mu}^* = -\frac{1}{2}\beta^{\#}(2\rho t + w_1^T).$$

Substitute $\tilde{\mu}^*$ into the second equation to get

$$2[\delta - \rho^T \beta^{\#} \rho]t^* - w_1 \beta^{\#} \rho + w_2 = 0$$

The coefficient of t^* is a negative number (or zero) since it is the Schur complement of δ in the matrix $\mathcal{R}$ which is negative semi-definite. We have

$$sign(t^*) = sign(w_2 - w_1 \beta^{\#} \rho)$$

Hence, $t^* > 0$ implies $w_2 - w_1\beta^{\#}\rho < 0$. Contradiction! Therefore $f(\tilde{\mu}, t_0) > 0 \; \forall t \geq 0$.

Necessity.

- We want to show that (a) (b) (c) or (d) holds. Suppose (a) (b) (c) all do not hold. Then, we need to show that (d) must hold. Since (a) (b) (c) do not hold, there exist μ_0, t_0 and $\eta_0 \neq 0$ such that $[\mu_0^T \; t_0 \; \eta_0]\Pi_f[\mu_0^T \; t_0 \; \eta_0]^T > 0$. This implies that $f(\mu_0/\eta_0, t_0/\eta_0) > 0$. Also, we must have $t_0/\eta_0 < 0$, since $f(\mu, t) \leq 0$, $\forall t \geq 0$. Therefore, a maximal value of the function f will be achieved in the region $t < 0$. That is, the partial derivatives vanish at some point (μ^*, t^*) with $t^* < 0$. As we showed in the proof of the sufficiency, $t^* < 0$ is equivalent to $w_2 - w_1\beta^{\#}\rho < 0$.

■