H^∞ CONTROL OF SYSTEMS WITH COMPENSATORS WHICH READ THE COMMAND SIGNAL *

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Abstract. This paper investigates H^∞ control design where the compensator is assumed to have immediate access to the command signal (usually denoted by W_1) but not to the disturbance signal (denoted by W_2). While the Doyle Glover paradigm only feeds a plant output Y into the compensator, in many industrial applications the command signal at time t can be transmitted to the compensator at time t if there is any advantage to doing so. In this article we analyze both linear and nonlinear systems and find that the advantage to this control architecture is great. The results in this paper generalize those of Ball *et al* 1993, Doyle *et al* 1988, Helton *et al* 1992. A necessary condition is given for the general nonlinear systems. These suggest formulas for controllers. For special cases such as: when there is no disturbance signal W_2 or the system is linear, we have very satisfactory conclusions. In particular, for linear systems with uncertainty, we derive conditions for the solution to the robust H^∞ control problem to exist.

Keywords: nonlinear systems, H^∞ control, output feedback, stabilization, uncertainty

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Section 1. INTRODUCTION

a. Problem statement

In many control situations the command signal W_1 could be transmitted directly to the compensator without delays or distortion. For example, major projects in San Diego are tokamak control and control of the Atlas Centaur launch vehicle. In tokamak control the signal W_1 specifies desired magnetic flux values at sensors around the plasma and Y is the values actually read at the sensors. The function W_1 is stored by the experimenter at the beginning of the experiment and so can be read at any time. In standard rocket models, W_1 is the command signal and is the first thing presented to the system. In addition to a command signal there are often disturbances modeled by W_2. A compensator which takes advantage of this produces the closed loop system in Figure 1.

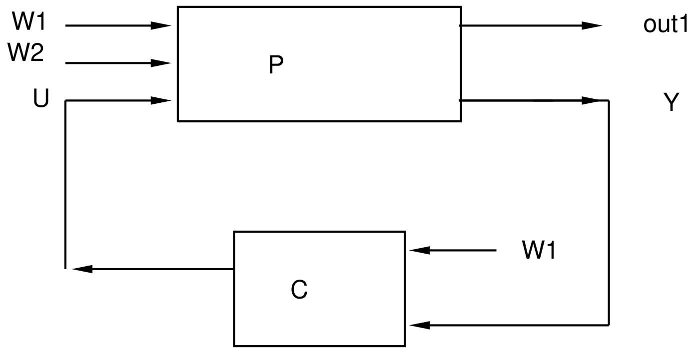

Figure 1.

The nonlinear input-affine (IA) plant is defined by

$$\begin{aligned} dx/dt &= A(x) + B_{11}(x)W_1 + B_{12}(x)W_2 + B_2(x)U \\ out &= C_1(x) + D_{12}(x)U \\ Y &= C_2(x) + D_{211}(x)W_1 + D_{212}(x)W_2 \end{aligned} \tag{1.1}$$

Here, W_1 stands for the command signal and W_2 stands for the disturbance signal, U is the control signal, *out* is the error signal, Y is the measurement signal, and $x \in \mathcal{X} = R^n$ is the state of the system. We wish to find a nonlinear time-invariant feedback compensator

$$dz/dt = a(z) + b_1(z)W_1 + b_2(z)Y, \quad U = c(z) \tag{1.2}$$

which improves performance. Such a compensator will be denoted by (a, b_1, b_2, c). We assume these systems are homogeneous throughout the entire paper, that is $A(0) = 0$, $C_1(0) = 0$, $C_2(0) = 0$, $a(0) = 0$, and $c(0) = 0$ so that $(0, 0)$ is an equilibrium point. We assume also that $D_{12}(x)^T D_{12}(x) = e_1(x) > 0$. The standard problem of H^∞ control in the nonlinear setting is:

(**IODISFBK**) *Find a stabilizing feedback law (a, b_1, b_2, c) so that the resulting closed loop system, starting from the origin (0,0), satisfies*

$$\int_0^{t_1} \|\mathrm{out}(t)\|^2 dt \le K \int_0^{t_1} \|\mathrm{W(t)}\|^2 dt \tag{1.3}$$

for a preassigned tolerance level K, where $W = (W_1^T, W_2^T)^T$.

Here stabilizing means that the closed loop system is asymptotically stable. In the terminology of Willems 1971 and Hill and Moylan 1980, the closed loop system is *dissipative* with respect to the particular energy supply rate $\varphi(W, \mathrm{out}) = K\|W\|_2^2 - \|\mathrm{out}\|_2^2$. Without loss of generality, We consider only the case $K = 1$.

Note that standard classical control configurations such as those with precompensation (see Figure 2) fit into this framework.

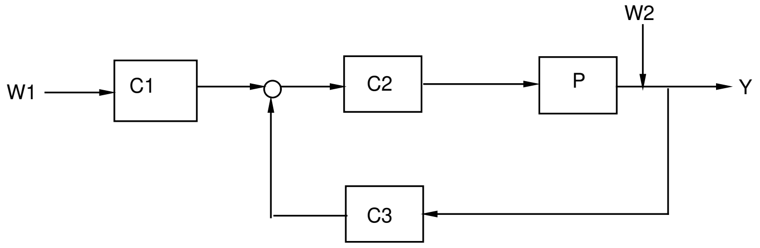

Figure 2.

Thus formulas presented here possibly will have considerable practical value. We begin by stating a special case of our main result. It answers the question for linear systems. Formulas for general nonlinear systems have a similar form, so this gives the flavor of results. However, for nonlinear systems the power of these formulas is less. Namely, while in the linear case the formulas are necessary and sufficient in the nonlinear case they are not in general. Much of the rest of the paper gives assumptions under which they hold.

b. Recipe for linear systems

Suppose the plant in Fig 1 is linear. Solve the algebraic Riccati inequalities $L_Y < L_X < 0$ for positive matrices $Y_I > X \geq 0$ where the Riccati expressions are

$$L_X = X(A - B_2 e_1^{-1} D_{12}^T C_1) + (A^T - C_1^T D_{12} e_1^{-1} B_2^T)X + C_1^T C_1$$

$$+X(B_{11}B_{11}^T + B_{12}B_{12}^T - B_2 e_1^{-1} B_2^T)X - C_1^T D_{12} e_1^{-1} D_{12}^T C_1 \tag{1.4}$$

$$L_Y = Y_I(A - B_{12} D_{212}^T e_3^{-1} C_2) + (A^T - C_2^T e_3^{-1} D_{212} B_{12}^T)Y_I + C_1^T C_1$$

$$+Y_I(B_{12}B_{12}^T - B_{12} D_{212}^T e_3^{-1} D_{212} B_{12}^T)Y_I - C_2^T e_3^{-1} C_2 \tag{1.5}$$

Choose $a(z) = a^* z$, $b_1(z) = b_1^*$, $b_2(z) = b_2^*$, and $c(z) = c^* z$ where

$$a^* = A + B_2 c^* - b_2 C_2 + \Big[(B_{11} - b_1 - b_2 D_{211})B_{11}^T + (B_{12} - b_2 D_{212})B_{12}^T\Big] X \tag{1.6}$$

$$c^* = -e_1^{-1}(B_2^T X + D_{12}^T C_1) \tag{1.7}$$

$$b_1^* = (Y_I - X)^{-1} Y_I B_{11} - b_2 D_{211} \tag{1.8}$$

$$b_2^* = (Y_I - X)^{-1}(C_2^T + Y_I B_{12} D_{212}^T) e_3^{-1/2} \tag{1.9}$$

Clearly here we need the assumptions $e_1 = D_{12}^T D_{12} > 0$ and $e_3 = D_{212} D_{212}^T > 0$.

Theorem 1.1. *If X and Y_I exist as above, then the recipe constructs a solution to the* **IODISFBK** *problem. Conversely, if a solution to the IODSFBK problem exists, then solutions X, Y_I exist with strict inequalities replaced by inequalities throughout.*

Proof: This theorem is an immediate consequence of more general theorems in the article. Formulas (1.6) and (1.7) specialize formulas (2.6) and (2.7) in section 2.b for nonlinear systems to linear systems and L_X is the linear version of IAX in section 2b. Theorem 3.1 establishes formulas (2.6) and (2.7) and in the process implies that (1.6) and (1.7) must give a compensator if the problem has a solution. Also it implies that X and Y_I with the properties in the recipe must exist. The other side of Theorem 1.1 is given as Corollary 4.2 as are the proofs of formulas (1.4) and (1.5) for b_1, b_2.

c. Comparison to classical H^∞ control.

Let us compare the conditions for the standard linear H^∞ control results and the results in this paper. It is easy to see that taking knowledge of the command signal into account produces an improvement in the achievable performance of the system.

Traditional linear H^∞ control is the same as we have here except that the compensator has no b_1 term (the term which reads the command signal W_1). In this traditional case sufficient conditions for linear H^∞ control are that certain algebraic Riccati expressions $IAYI(x) < IAX(x) < 0$ (cf. **[2]** **[8]**) have matrix solutions $Y_I > X > 0$ and that these conditions with nonstrict inequality are necessary.

The solution when the compensator has a b_1 term is governed by Theorem 1.1. The performance $K < 1$ is achieved when there exists $Y_I > X > 0$ solving $L_Y < L_X < 0$. For easy comparison, we suppose that the following simplifying assumptions are satisfied:

$$B_{11} D_{211}^T = B_{12} D_{212}^T = 0.$$

One can check that $IAX(x) = x^T L_X x$. Therefore, to show that our design can achieve better performance than the standard linear H^∞ optimal design, we only need to show that $IAYI \geq x^T L_Y x$. Indeed, one can check that for any matrix Y_I

$$IAYI(x) - x^T L_Y x = x^T [Y_I (B_{11} B_{11}^T) Y_I + C_2^T (e_3^{-1} - e_2^{-1}) C_2] x,$$

where

$$e_2 = D_{211} D_{211}^T + D_{212} D_{212}^T = D_{211} D_{211}^T + e_3 \geq e_3.$$

Thus $e_2^{-1} \leq e_3^{-1}$, and $IAYI(x) - x^T L_Y x \geq 0$. We conclude any Y_I which satisfies the H^∞ control conditions with b_1 (most critically $IAYI \leq 0$) also satisfies $L_Y \leq 0$ with room to spare. This spare positivity translates directly into improved performance.

More precise comparisons of improvement arise from the formula:

$$IAYI(x) - x^T L_Y x = x^T [Y_I (B_{11} B_{11}^T) Y_I + C_2^T e_2 \delta^{1/2} (1 - \delta^{1/2} e_2 \delta^{1/2})^{-1} \delta^{1/2} e_2 C_2] x$$

where $\delta = D_{211} D_{211}^T$.

Section 2. FORMAL SOLUTION FOR NONLINEAR SYSTEMS

a. Another control problem

Our first step is to state a control problem, denoted ε-**DISFBK**, which is the state space version of (**IODISFBK**). While these two problems are not fully equivalent in the nonlinear case actually ε-**DISFBK** is the one we solve. In the linear case this problem and **IODISFBK** are equivalent so a common approach to H^∞ control is to solve ε-**DISFBK**. That is how the results in the earlier section are derived.

The version of the H^∞ control problem (see Van der Shaft 1991, Isidori and Astolfi 1992, Ball *etal* 1993.) which we shall treat in this paper is:

($\varepsilon-$ **DISFBK**) *Find a non-negative differentiable function* ε *on* $\mathcal{X} \times \mathcal{X}$ *with* $\varepsilon(0,0) = 0$ *so that* *there exist $a(z), b_1(z), b_2(z)$, and $c(z)$ which satisfy the well known dissipation inequality*

$$0 \geq \max_{x,z,W} H_{a(z),b_1(z),b_2(z),c(z)}(W,x,z), \tag{2.1}$$

where

$$\begin{aligned} H_{a,b_1,b_2,c}(W,x,z) &= -W_1^T W_1 - W_2^T W_2 + \|C_1(x) + D_{12}(x)c\|^2 \\ &+ \nabla_x \varepsilon(x,z)^T \Big[A(x) + B_{11}(x)W_1 + B_{12}(x)W_2 + B_2(x)c\Big] \\ &+ \nabla_z \varepsilon(x,z)^T \Big\{a + b_1 W_1 + b_2 \Big[C_2(x) + D_{211}(x)W_1 + D_{212}(x)W_2\Big]\Big\} \end{aligned} \tag{2.2}$$

Also we wish to find formulas for or properties of a, b_1, b_2, and c.

We refer to the above statement as the $\varepsilon-$**DISFBK** problem. These two problems are related through the notion of an *energy* or *storage function.* A storage or energy function of a system

$$\dot{x} = F(x,W),\ out = G(x,W) \tag{2.3}$$

is a nonnegative function ε on the state space $\mathcal{X}$ satisfying

$$\int_{t_0}^{t_1} \{\|\mathrm{out}(t)\|^2 - \|W(t)\|^2\} dt \leq \varepsilon(x(t_0)) - \varepsilon(x(t_1)) \tag{2.4}$$

and $\varepsilon(0) = 0$. If ε is differentiable, define the energy Hamiltonian $\mathcal{H}$ for the same system by

$$\mathcal{H} = \|out\|^2 - \|W\|^2 + \nabla\varepsilon(x)^T F(x,W) \tag{2.5}$$

Proposition 2.1.

(1) The system (2.3) satisfies the inequality (1.3) with $K = 1$ if and only if there exists a storage function ε on $\mathcal{X}$ and $0 \leq \varepsilon(x) < \infty$ for $x \in \mathcal{X}$ reachable from 0.

(2) Assume that a function $\varepsilon(x) \geq 0$ is differentiable for $x \in \mathcal{X}$ reachable from 0 with $\varepsilon(0) = 0$. Then ε is a storage function for the system (2.3) if and only if the inequality $\mathcal{H} \leq 0$ holds for all reachable $x \in \mathcal{X}$.

Proof: (1) See Hill and Moylan 1980, Theorem 4. (2) See Ball *et al* 1993, Theorem 3.1. ∎

For a detailed discussion of the relationship between **IODISFBK** and $\varepsilon-$ **DISFBK** problems, the interested reader is referred to Ball *et al* 1993.

A subtlety is that the $\varepsilon-$ **DISFBK** condition or even the sought after inequality (1.3) does not gaurantee stability of the closed loop system. It comes close in that if the $\varepsilon-$ **DISFBK** conditions (2.1) and ε strictly positive hold then the closed loop system is stable as required by **IODISFBK**. For linear H^∞ control $\varepsilon-$ **DISFBK** can be used to obtain stablizing solutions provided one insists that the famous X, Y solutions one uses are stablizing solutions to the Riccati equations they solve. The implications of this rather subtle point is discussed in Helton and James 1994.

In the next subsection we propose a class of recipes for the solution to ε-**DISFBK** problem. In Section 3 it is proved that the recipes are the only possible solutions with compensators whose state space has the same dimension as the state space of the plant. In Section 4 we investigate the ε-**DISFBK** problem for linear systems with time-varying norm-bounded uncertainty. Necessary and sufficient conditions are given in Section 5 for nonlinear systems when there is no disturbance signal W_2. The main results in Sections 1, 2 and 3 were announced for a special case where there is no noise signal W_2 in Helton *et al* 1992.

b. Formal Recipe for Solution

We begin with basic definitions. Unless explicitly pointed out we only consider compensators with $dim(z) = dim(x)$. This is the most stringent assumption in the paper. Although in traditional H^∞ control it is automatically satisfied so imposes no restriction (cf. Doyle *et al* 1988).

Let $\mathcal{D}$ denote the diagonal $\{(x,z) : x = z\}$ of $\mathcal{X} \times \mathcal{X}$. As seen in Ball *et al* 1993, the key set is $Z_\varepsilon = \{(x,z) : \nabla_z \varepsilon(x,z) = 0\}$. As we do in Ball *et al* 1993, we use the fact that the unknowns a, b_1, b_2, c often have a free diffeomorphism φ on the state space to normalize Z_ε to contain the diagonal $\mathcal{D}$. Hereafter, we assume that $Z_\varepsilon \supseteq \mathcal{D}$. Define the nonlinear analogue of the

"Doyle-Glover Riccati Operator" (see, e.g. Doyle *et al* 1988) as

$$
\begin{aligned}
k(x,z) = 2\Big\{ &\Big[C_1(z)^T D_{12}(z) e_1(z)^{-1} e_1(x) - C_1(x)^T D_{12}(x)\Big] e_1(z)^{-1} B_2(z)^T \\
&+ \Big[C_2(z)^T - C_2(x)^T\Big] e_3(x)^{-1} \Big[[D_{211}(x) - D_{211}(z)] B_{11}(z)^T - D_{212}(z) B_{12}(z)^T\Big] \Big\} X(z) \\
&+ p^T \Big\{ A(x) - B_{12}(x) D_{212}(x)^T e_3(x)^{-1} \Big[C_2(x) - C_2(z)\Big] - B_2(x) e_1(z)^{-1} D_{12}(z)^T C_1(z) \Big\} \\
&+ q^T \Big\{ A(z) - B_2(z) e_1(z) D_{12}(z)^T C_1(z) \Big\} + p^T B_{12}(x) \Big[I - D_{212}(x)^T e_3(x)^{-1} D_{212}(x)\Big] B_{12}(x)^T p/4 \\
&+ p^T \Big\{ B_{11}(x) B_{11}(z)^T - B_{12}(x) D_{212}(x)^T e_3(x)^{-1} [D_{211}(x) - D_{211}(z)] B_{11}(z)^T \\
&- B_2(x) e_1(z)^{-1} B_2(z)^T + B_{12}(x) D_{212}(x)^T e_3(x)^{-1} D_{212}(z) B_{12}(z)^T \Big\} X(z) + C_1(x)^T C_1(x) \\
&+ q^T \Big\{ B_{11}(z) B_{11}(z)^T + B_{12}(z) B_{12}(z)^T - B_2(z) e_1(z)^{-1} B_2(z)^T \Big\} X(z) \\
&+ X(z)^T \Big\{ \Big[2B_{12}(z) D_{212}(z)^T + B_{11}(z) D_{211}(z)^T - B_{11}(z) D_{211}(x)^T\Big] e_3(x)^{-1} [D_{211}(x) \\
&- D_{211}(z)] B_{11}(z)^T - B_{12}(z) D_{212}(z)^T e_3(x)^{-1} D_{212}(z) B_{12}(z)^T - B_{11}(z) B_{11}(z)^T \\
&+ B_2(z) e_1(z)^{-1} e_1(x) e_1(z)^{-1} B_2(z)^T \Big\} X(z) - \Big[C_2(x)^T - C_2(z)^T\Big] e_3(x)^{-1} \Big[C_2(x) - C_2(z)\Big] \\
&+ C_1(z)^T D_{12}(z) e_1(z)^{-1} \Big[e_1(x) e_1(z)^{-1} D_{12}(z)^T C_1(z) - 2 D_{12}(x)^T C_1(z)\Big]
\end{aligned}
$$

where $p = \nabla_x \varepsilon(x,z)$, $q = \nabla_z \varepsilon(x,z)$, $X(z) = \frac{1}{2}\nabla_x \varepsilon(x,x)$, $e_3(x) = D_{212}(x) D_{212}(x)^T > 0$. In particular, we are interested in the function $k(x,x)$, which we denote, following the convention of Ball *et al* 1993, by $IAX(x) := k(x,x)$. Thus

$$
\begin{aligned}
IAX(x) = {} & 2\Big[A(z)^T - C_1(z)^T D_{12}(z) e_1(z)^{-1} B_2(z)^T\Big] X(z) \\
& + X(z)^T \Big[B_{11}(z) B_{11}(z)^T + B_{12}(z) B_{12}(z)^T - B_2(z) e_1(z)^{-1} B_2(z)^T\Big] X(z) \\
& - C_1(z)^T \Big[D_{12}(z) e_1(z)^{-1} D_{12}(z)^T - I\Big] C_1(z).
\end{aligned}
$$

Note that the function $IAX(x)$ is well-defined even when $e_3(x)$ is not positive definite which is necessary for the definition of $k(x,z)$.

Recipe: Find a solution X for $IAX(x) \leq 0$ with $X(x) = \frac{1}{2}\nabla_x \varepsilon(x,x)$ and $\varepsilon(x,z)$ positive definite (positive definite means $\varepsilon(x,z) > 0$ for $(x,z) \neq 0$ and $\varepsilon(0,0) = 0$). Choose a controller (1.2) with $a(z) = a^*(z)$, $c(z) = c^*(z)$ given by

$$a^*(z) = A(z) + B_2(z)c^*(z) + \Big\{B_{11}(z)B_{11}(z)^T + B_{12}(z)B_{12}(z)^T - b_1(z)B_{11}(z)^T$$

$$-b_2(z)D_{211}(z)B_{11}(z)^T - b_2(z)D_{212}(z)B_{12}(z)^T\Big\}X(z) - b_2(z)C_2(z) \tag{2.6}$$

$$c^*(z) = -e_1(z)^{-1}\Big[B_2(z)^T X(z) + D_{12}(z)^T C_1(z)\Big] \tag{2.7}$$

In general, how to choose $b_1(z)$, $b_2(z)$ is still an open problem. We will discuss some special cases in Section 4 and Section 5 where we do have formulas for $b_1(z)$ and $b_2(z)$. The known properties of the recipe are the subject of the next section.

Section 3. THE RECIPE IS NECESSARY

In this section we show that there is less flexibility than one might imagine in the Recipe. We do this by giving appealing conditions under which the formulas in the recipe are the only possible solution to the problem.

Theorem 3.1. *Assume there exists a compensator* (a, b_1, b_2, c) *and an energy function* ε *satisfying* $Z_\varepsilon \supseteq \mathcal{D}$ *which solves* ε**-DISFBK**. *Then*

(1) $IAX(x) \leq 0$ *has a solution* $X(x)$ *which is the gradient of a function* $\eta(x) \geq 0$ *with* $\eta(0) = 0$.

(2) If $X(x)$ *actually satisfies* $IAX(x) = 0$ *and if* $D_z(\nabla_z \varepsilon(x,z))\big|_{z=x}$ *has full rank, then a separation principle holds. That is,* c *must be given by (2.7) and* a *must be given by (2.6). If in addition,* $e_3(x) > 0$ *then we must have* $k(x,z) \leq 0$.

Proof: We shall need the following definition.

$$H^{*W}_{a,b_1,b_2,c}(x,z) := \max_W H_{a,b_1,b_2,c}(W,x,z) \tag{3.1}$$

For $H_{a,b_1,b_2,c}$ in (2.2) we find that the maximizing W is given by

$$W_1^* = B_{11}(x)^T \nabla_x \varepsilon(x,z)/2 + b_1(z)^T \nabla_z \varepsilon(x,z)/2 + D_{211}(x)^T b_2(z)^T \nabla_z \varepsilon(x,z)/2,$$

$$W_2^* = B_{12}(x)^T \nabla_x \varepsilon(x,z)/2 + D_{212}(x)^T b_2(z)^T \nabla_z \varepsilon(x,z)/2.$$

Clearly, the dissipation inequality (2.1) holds for a particular compensator $(a(z), b_1(z), b_2(z), c(z))$ if and only if $H^{*W}_{a(z),b_1(z),b_2(z),c(z)}(x,z) \leq 0$ for all x, z. Hereafter we abbreviate $H^{*W}_{a,b_1,b_2,c}(x,z)$ by H^{*W} if it is clear from the context what a, b_1, b_2, c are.

(1) A successful compensator and energy function ε must satisfy

$$H^{*W}_{a(z),b_1(z),b_2(z),c(z)}(x,z) \leq 0$$

for all x, z. In particular, for $x = z$, the function $H^{*W}_{a,b_1,b_2,c}(x,z)$ is independent of a, b_1 and b_2 and

$$IAX(x) = \min_c H^{*W}_{a(x),b_1(x),b_2(x),c}(x,x) \leq H^{*W}_{a(x),b_1(x),b_2(x),c(x)}(x,x) \leq 0$$

for all x. Here the minimizing c is given by (2.7), with $X(x) = \nabla \eta(x)$, and $\eta(x) = \frac{1}{2}\varepsilon(x,x)$.

(2) It is easy to see that $H^{*W}_{a,b_1,b_2,c}(x,x)$, as a function of c, has the following form

$$H^{*W}_{a,b_1,b_2,c}(x,x) = f_1 + f_2 c + c^T e_1(x) c.$$

Since $e_1(x) > 0$ the above function has a unique minimum value. From the proof of (1), the minimizing c is giving by $c^*(x)$. Thus if $c \neq c^*(x)$, then for some x

$$H^{*W}_{a,b_1,b_2,c}(x,x) > H^{*W}_{a,b_1,b_2,c^*}(x,x) = 0.$$

This contradicts our assumption. Therefore we must have $c(z) = c^*(z)$.

For the derivation of $a^*(z)$ we use the Separation Principle in Ball *et al* 1993. Since $H^{*W}|_{\mathcal{D}} = 0$ and $H^{*W}(x,z) \leq 0$, H^{*W}, as a function of z, achieves local maximum at $z = x$. Thus we must have $D_z H^{*W}(x,z)|_{z=x} = 0$. ¿From this equation we can solve for $a(z)$. All calculations were done using the Mathematica software package called NCAlgebra (available upon request from NCALG@osiris.ucsd.edu). As a result we get the formulas (2.6) and (2.7).

(3) Plug these formulas for a and c into H^{*W} to get

$$H^{*W} = k(x,z) + \|b_{12}term\|^2 + \|b_2 term\|^2, \quad where$$

$$b_{12}term = q^T b_1(z) + q^T b_2(z) D_{211}(x) + p^T B_{11}(x) - 2\,X(z)^T B_{11}(z),$$

$$b_2 term = \sqrt{e_3(x)} b_2(z)^T q + 2\Big[C_2(x) - C_2(z)\Big] + D_{212}(x) B_{12}(x)^T p$$

$$+2\Big\{\Big[D_{211}(x) - D_{211}(z)\Big] B_{11}(z)^T - D_{212}(z) B_{12}(z)^T\Big\} X(z),$$

$p = \nabla_x \varepsilon(x,z)$, and $q = \nabla_z \varepsilon(x,z)$. In order for H^{*W} to be nonpositive, it is necessary that $k(x,z) \le 0$. The assumption $e_3(x) > 0$ is needed to complete the squares for H^{*W}. ∎

Remark. Note that even if $k(x,x) \neq 0$ but is merely ≤ 0, a^* and c^* in (2.6) and (2.7) are still reasonable candidates for a solution.

In general we do not have necessary and sufficient conditions, and we do not know how to pick $b_1(z)$, and $b_2(z)$. But in some special cases we have quite satisfactory results. These special cases are 1) The system is linear; 2) The disturbance signal W_2 is not present (part of the results can be found in Helton *et al* 1992). The case when the command signal W_1 is not present was discussed in Ball *et al* 1993. This paper is a generalization of Ball *et al* 1993 and Helton *et al* 1992.

Section 4. LINEAR SYSTEMS WITH UNCERTAINTY

In this section we consider the H^∞ control problem for linear systems and simultaneously we treat plant uncertainty. We tackle the problem with two different approaches. The first approach is based on diagonalizing the Hamiltonian (Lemma 4.1). It eventually leads to some sufficient conditions and some necessary conditions (stated in Theorem 4.3). Also in the course of this section we develop the linear theory. The second approach applies to more special systems and is based on comparing an uncertain system and a parametrized nominal system and gives necessary and sufficient conditions (stated in Theorem 4.7). Although the comparison approach gives better specialized results, we introduce both approaches because the comparison approach is difficult to extend to more general cases. On the contrary, the Hamiltonian diagonalization approach appears to be more powerful in generalizing to different cases. Also for the comparison approach we need some of the results developed in the first approach.

The system we consider has the following form

$$\begin{aligned} dx/dt &= (A + \Delta A)x + B_{11}W_1 + B_{12}W_2 + B_2U \\ out &= C_1x + D_{12}U \\ Y &= C_2x + D_{211}W_1 + D_{212}W_2 \end{aligned} \tag{4.1a}$$

where ΔA is the uncertainty in the system which satisfies the following constraint

$$\Delta A = DF(t)E, \qquad F(t)^TF(t) \leq I \tag{4.1b}$$

for some known constant matrices E and D, and the elements in $F(t)$ are Lebesgue-measurable. We consider the dynamic compensator of the following form:

$$dz/dt = az + b_1W_1 + b_2Y, \quad U = cz \tag{4.2}$$

Without other information on the uncertainty of the system it is reasonable to use the a, c in (1.6-1.7) as we do in the no uncertainty case, where X, Y_I may be subject to more restrictive constraints. We will study when this kind of compensator works for the whole class of systems. To be more specific, we formulate the following problem

(**Robust-DISFBK**)

Find a positive definite quadratic energy function $\varepsilon(x,z)$, *with* $\varepsilon(0,0)=0$ *and constant matrices* a, b_1, b_2, *and* c *so that for the closed loop system (4.1-4.2) the dissipation inequality*

$$0 > \max_{x,z,W} H_{a(z),b_1(z),b_2(z),c(z)}(W,x,z),$$

(the strict inequality version of (2.1)) holds for all uncertainties satisfying (4.1b).

Note that if the **Robust-DISFBK** problem is solvable then by choosing the energy function as the Lyapunov function one can show that the unforced closed loop system is asymptotically stable. With the choice of a, c given by (1.6-1.7) we can easily calculate H^{*W} for the closed loop system. To emphasize that the Hamiltonian H^{*W} contains the uncertain element ΔA we denote it by $H^{*W}(\Delta A)$. Now we describe the Hamiltonian diagonalization approach.

a. Diagonalizing the Hamiltonian

Lemma 4.1. *Suppose* $e_3 = D_{212}D_{212}^T > 0$. *The* **Robust-DISFBK** *problem for the system (4.1) is solvable with* a, c *given by (1.6-1.7) if and only if there exist* $Y_I > X > 0$ *such that*

$$\tilde{L}_X := L_X + X\Delta A + \Delta A^T X < 0 \tag{4.3}$$

$$L_Y - L_X + (Y_I - X)[\Delta A(-\tilde{L}_X)^{-1}\Delta A^T](Y_I - X) < 0 \tag{4.4}$$

for all ΔA *satisfying (4.1b). Here* L_X *and* L_Y *are defined by (1.4) and (1.5). If (4.3) (4.4) hold with* $Y_I > X > 0$, *then* b_1, b_2 *for a suitable controller are given by (1.8)(1.9).*

Proof: Let us first consider the form of the energy function. For the sufficient part of the proof we construct the energy function to be

$$\varepsilon(x,z) = x^T X x + (x-z)^T (Y_I - X)(x-z).$$

Since in the definition of **Robust-DISFBK** we assume that the energy function is quadratic, and throughout the paper we assume $\nabla_z \varepsilon(x,z) = 0$ for $z = x$, it is also necessary that the energy function takes the above form.

Plug this and a^*, c^* into $H^{*W}(\Delta A)$. As in Theorem 3.1 it has the form

$$H^{*W} = k(x,z) + \|b_{12}term\|^2 + \|b_2 term\|^2$$

where in the linear case b_1^*, b_2^* in (1.8), (1.9) are exactly what make $b_{12}term = 0$,and $b_2 term = 0$; thus they minimize H^{*W}. With these choices for a, b_1, b_2, c we have

$$H^{*W}(\Delta A) = k(x,z) = [\, x^T \quad z^T \,]\, H \begin{bmatrix} x \\ z \end{bmatrix}$$

where

$$H = \begin{bmatrix} L_Y + Y_I \Delta A + \Delta A^T Y_I & L_X - L_Y - \Delta A^T (Y_I - X) \\ L_X - L_Y - (Y_I - X)\Delta A & L_Y - L_X \end{bmatrix}$$

Note that when $x = z$ we have $H^{*W}(\Delta A) = x^T \tilde{L}_X x$. Therefore in order to have $H^{*W}(\Delta A) < 0$ it is necessary that $\tilde{L}_X$ is nonsingular. Let

$$S = \begin{bmatrix} I & \tilde{L}_X^{-1} \Delta A^T (Y_I - X) \\ I & I + \tilde{L}_X^{-1} \Delta A^T (Y_I - X) \end{bmatrix}$$

We have $det(S) \neq 0$ and

$$H_1 = S^T H S = \begin{bmatrix} \tilde{L}_X & 0 \\ 0 & L_Y - L_X - (Y_I - X)[\Delta A \tilde{L}_X^{-1} \Delta A^T](Y_I - X) \end{bmatrix}$$

Therefore $H^{*W}(\Delta A) < 0$ if and only if $H_1 < 0$, which is clearly equivalent to (4.3) and (4.4).

■

Remark From the definition of the uncertainty ΔA it is clear that all the possible uncertainties form a compact set. Therefore (4.3) is equivalent to $\tilde{L}_X < -p_0 I$ for some positive number p_0. We will use this fact later in the proof of Theorem 4.3.

Corollary 4.2. *Suppose* $e_3 = D_{212}D_{212}^T > 0$. *For linear systems (4.1) with no uncertainty, i.e.* $\Delta A = 0$, *there exists a positive definite quadratic energy function which, together with the* a, c *given by (1.6) and (1.7), makes* $H^{*W} < 0$ *if and only if there exists* $Y_I > X > 0$ *solving* $L_Y < L_X < 0$. *The formulas for* b_1 *and* b_2 *given in (1.8) (1.9) provide a solution if a solution exists.*

Proof: This is just a direct application of Lemma 4.1 with $\Delta A = 0$. ■

Note that in Lemma 4.1 and Corollary 4.2 the closed loop systems are asymptotically stable since $\varepsilon(x,z) > 0$ and $\dot{\varepsilon}(x,z) < 0$. When there is no command signal available for feedback, the conditions in Corollary 4.2 coincide with the standard Doyle-Glover-Khargonekar-Francis result.

The following theorem gives some necessary conditions and sufficient conditions which do not involve uncertain elements; thus they are particularly easy to check.

Theorem 4.3. *The* **Robust-DISFBK** *problem is solvable for the system (4.1a) with uncertainty* ΔA *satisfying (4.1b) if there exist* $Y_I > X > 0, \epsilon > 0, p_0 > 0, p_1 > 0$ *such that* $\frac{||E||^2||D||^2}{p_0} \le p_1$,

$$L_X + \epsilon X D D^T X + \frac{1}{\epsilon} E^T E + p_0 I < 0 \tag{4.5}$$

$$L_Y - L_X + p_1 (Y_I - X)^2 < 0. \tag{4.6}$$

In this case we can choose a, b_1, b_2, c *to be given by (1.6-1.9). Conversely, if the* **Robust-DISFBK** *problem is solvable with* a, c *given by (1.6) and (1.7) for the system (4.1a) with uncertainty* ΔA *satisfying (4.1b) then there exist* $Y_I > X > 0, \epsilon > 0, p_0 > 0, p_1 > 0$ *such that (4.5) and (4.6) hold.*

Proof: First let us prove that (4.3) is equivalent to (4.5) (also see Shi *et al* 1992).

(4.5) => (4.3): Using the inequality $M^T N + N^T M \le \epsilon M^T M + \frac{1}{\epsilon} N^T N$ we have

$$L_X + XDF(t)E + E^T F(t)^T D^T X \le L_X + \epsilon X D D^T X + \frac{1}{\epsilon} E^T E < -p_0 I.$$

(4.3) <= (4.5): For all $F(t)$ we have $F(t)^T F(t) \leq I$ and $x \neq 0$ and so

$$x^T(L_X + p_0 I)x < -2\, x^T X D F(t) E x.$$

Hence

$$x^T(L_X + p_0 I)x < -2\, max\{x^T X D F(t) E x : F(t)^T F(t) \leq I\} \leq 0.$$

Using Lemma A.2 of Khargonekar *et al* 1990, we have

$$[x^T(L_X + p_0 I)x]^2 > 4\, max\{x^T X D F(t) E x : F(t)^T F(t) \leq I\}$$

$$= 4\, x^T X D D^T X x x^T E^T E x.$$

It follows from Lemma A.3 of Khargonekar *et al* 1990 that (4.5) holds.

Suppose (4.4) holds for all ΔA, in particular for $\Delta A = 0$, in which case we have $L_Y - L_X < 0$. Since L_Y, L_X, Y_I, and X are all time-invariant, there exists sufficiently small $p_1 > 0$ such that (4.6) holds.

Suppose (4.5) and (4.6) hold, and $\|E\|^2 \|D\|^2 \leq p_0 p_1$. Then $\tilde{L}_X < -p_0 I$ from above, or equivalently, $(-\tilde{L}_X)^{-1} < \frac{1}{p_0} I$. Thus

$$\text{Left Hand Side of (4.4)} \leq L_Y - L_X + (Y_I - X)[\Delta A \frac{I}{p_0} \Delta A](Y_I - X)$$

$$= L_Y - L_X + (Y_I - X)[D F(t) \frac{E E^T}{p_0} F(t)^T D^T](Y_I - X)$$

$$\leq L_Y - L_X + (Y_I - X)[D F(t) F(t)^T D^T \frac{p_1}{\|D\|^2}](Y_I - X)$$

$$\leq L_Y - L_X + (Y_I - X)\Big[p_1 \frac{D D^T}{\|D\|^2}\Big](Y_I - X)$$

$$\leq L_Y - L_X + p_1 (Y_I - X)(Y_I - X) < 0$$

■

b. Comparison Approach

Next let us investigate this problem with a different approach, which ultimately yields the main results Theorem 4.7. This technique was first used in Xie *et al* 1992 for the special case where there is no command signal W_1. Using Corollary 4.2 we are able to extended the result in Xie *et al* 1992 to the more general case. Consider the following two systems:

$$\begin{aligned} dx/dt &= (A + \Delta A)x + B_{11}W_1 + B_{12}W_2 \\ out &= C_1 x + D_{12}U \\ Y &= C_2 x + D_{211}W_1 + D_{212}W_2 \end{aligned} \tag{4.7}$$

$$\begin{aligned} dx/dt &= Ax + B_{11}W_1 + B_{12}W_2 + \sqrt{\epsilon}DW_3 \\ out_1 &= C_1 x + D_{12}U \\ out_2 &= \frac{1}{\sqrt{\epsilon}}Ex \\ Y &= C_2 x + D_{211}W_1 + D_{212}W_2 \end{aligned} \tag{4.8}$$

where ΔA is the system uncertainty satisfying (4.1b). We have the following basic lemma.

Lemma 4.6. *The system (4.7) has an* H^∞ *gain less than or equal to 1 and is asymptotically stable (with* $W_1 = 0, W_2 = 0$*) if and only if the system (4.8) has an* H^∞ *gain less than or equal to 1 and is asymptotically stable (with* $W_1 = 0, W_2 = 0, W_3 = 0$*).*

Proof: With some slight change of notation the proof for the main result in Xie *et al* 1992 also works here. ∎

Using this fact we derive the following

Theorem 4.7. *Suppose* $dim(z) = dim(x)$, $Z_\varepsilon = \mathcal{D}$ *and* $e_3 > 0$. *Then the* **Robust-DISFBK** *problem is solvable, with* a, c *given by (1.6) and (1.7), for the system (4.1) if and only if there exists*

$\epsilon > 0$ such that $L_{Ya} < L_{Xa} < 0$ has positive definite solutions $Y_I > X > 0$, where $L_{Ya}(L_{Xa})$ is the $L_Y(L_X)$ for the auxiliary system (4.8), i.e.

$$L_{Ya} = Y_I(A - B_{12}D_{212}^T e_3^{-1}C_2) + (A^T - C_2^T e_3^{-1} D_{212} B_{12}^T)Y_I + C_1^T C_1$$

$$+Y_I(B_{12}B_{12}^T - B_{12}D_{212}^T e_3^{-1} D_{212} B_{12}^T)Y_I - C_2^T e_3^{-1} C_2 + \frac{1}{\epsilon}E^T E + \epsilon Y_I D D^T Y_I \tag{4.9}$$

$$L_{Xa} := X(A - B_2 e_1^{-1} D_{12}^T C_1) + (A^T - C_1^T D_{12} e_1^{-1} B_2^T)X + C_1^T C_1$$

$$+X(B_{11}B_{11}^T + B_{12}B_{12}^T - B_2 e_1^{-1} B_2^T)X - C_1^T D_{12} e_1^{-1} D_{12}^T C_1 + \frac{1}{\epsilon}E^T E + \epsilon X D D^T X \tag{4.10}$$

Proof: Consider the system (4.1) with a compensator (4.2). The closed loop system can be written as

$$d\eta/dt = \left[\begin{bmatrix} A & B_2 c \\ b_2 C_2 & a \end{bmatrix} + \begin{bmatrix} \Delta A & 0 \\ 0 & 0 \end{bmatrix}\right]\eta + \begin{bmatrix} B_{11} & B_{12} \\ b_2 D_{211} + b_1 & b_2 D_{212} \end{bmatrix} W$$
$$out = [\,C_1 \quad D_{12}c\,]\eta$$

Let

$$\Delta A' := \begin{bmatrix} \Delta A & 0 \\ 0 & 0 \end{bmatrix} \quad D' := \begin{bmatrix} D \\ 0 \end{bmatrix}, \quad E' := [\,E \quad 0\,]$$

Then from Lemma 4.6 the H^∞ gain is less than or equal to 1 if and only if there exists $\epsilon > 0$ such that the H^∞ gain of the system

$$d\eta/dt = \begin{bmatrix} A & B_2 c \\ b_2 C_2 & a \end{bmatrix}\eta + \begin{bmatrix} B_{11} & B_{12} & \sqrt{\epsilon}D \\ b_2 D_{211} + b_1 & b_2 D_{212} & 0 \end{bmatrix} W'$$
$$out_1 = [\,C_1 \quad D_{12}c\,]\eta$$
$$out_2 = [\,\tfrac{1}{\sqrt{\epsilon}}E \quad 0\,]\eta$$

is less than or equal to 1. Here $W' = (W^T, W_3^T)^T$. But this system can be obtained from the following system

$$dx/dt = Ax + B_{11}W_1 + B_{12}W_2 + \sqrt{\epsilon}DW_3 + B_2 U$$

$$\begin{aligned} out_1 &= C_1 x + D_{12} U \\ out_2 &= \frac{1}{\sqrt{\epsilon}} E x \\ Y &= C_2 x + D_{211} W_1 + D_{212} W_2 \end{aligned} \tag{4.11}$$

by applying the same dynamic feedback (4.2).

Thus the necessary and sufficient conditions for the **IODISFBK** problem for the system (4.1) reduce to the conditions for the system (4.11) which can be derived from Corollary 4.2. ■

Although for some special cases one can extend the result in Theorem 4.7 to the case when there are uncertainties on the other matrices such as B_1, B_2, C_1, and C_2, in general this approach does not work. It is our hope that these results could be extended to the case where there are other type of uncertainties in the system using the first approach which yields Theorem 4.3.

Lemma 4.1 and Theorem 4.3 can be easily modified for the case where there is no disturbance signal ($e_3 = 0$). Some results were given in Helton *et al* 1992 without proof. An interested reader can derive the formulas for that case as an exercise.

Section 5. W_2 IS NOT PRESENT

In this section we shall discuss the special case where there is no disturbance signal W_2 and the command signal W_1 is accessible. The system is defined to be

$$\begin{aligned} dx/dt &= A(x) + B_1(x) W_1 + B_2(x) U \\ out &= C_1(x) + D_{12}(x) U \\ Y &= C_2(x) + D_{21}(x) W_1 \end{aligned} \tag{5.1}$$

The compensator has the following form

$$dz/dt = a(z) + b_1(z) W_1 + b_2(z) Y, \quad U = c(z). \tag{5.2}$$

a. Reduction of ε-DISFBK to a stabilization problem

The formal recipe of section 2 when specialized to this case is:

$$a^*(z) = A(z) + B_2(z)c^*(z) + \Big[B_1(z) - b_1(z) - b_2(z)D_{21}(z)\Big]B_1(z)^T X(z) - b_2(z)C_2(z) \tag{5.3}$$

$$c^*(z) = -e_1(z)^{-1}\Big[B_2(z)^T X(z) + D_{12}(z)^T C_1(z)\Big] \tag{5.4}$$

Proposition 5.1. *Assume that* $a(z)$ *is given by (5.3) and* $c(z)$ *by (5.4). Set*

$$b_1(z) = B_1(z) - b_2(z)D_{21}(z). \tag{5.5}$$

Then the compensator takes the form of a state estimator followed by an H^∞ *optimal state feedback. That is, the closed loop system is*

$$\begin{aligned} \dot{x} =& A(x) + B_1(x)W_1 - B_2(x)e_1(z)^{-1}\{B_2(z)^T X(z) + D_{12}(z)^T C_1(z)\} \\ \dot{z} =& A(z) + B_1(z)W_1 - B_2(z)e_1(z)^{-1}\{B_2(z)^T X(z) + D_{12}(z)^T C_1(z)\} \\ &+ b_2(z)\{C_2(x) + D_{21}(x)W_1 - [C_2(z) + D_{21}(z)W_1]\} \end{aligned} \tag{5.6}$$

Proof: Substitute the formulas for a, b_1, c into (5.1), (5.2). We have the closed loop system (5.6). It is easy to see that the z equation consists of a copy of the x equation and an adjusting term based on the output error. This is clearly the structure of the state estimator. The feedback $U = c(z)$ takes the form of optimal state feedback with the state x replaced by z. ■

Theorem 5.2. *Assume there exists* $X(x) = \nabla\eta(x), \eta \geq 0, \eta(0) = 0$ *solving* $IAX(x) \leq 0$ *and* $b_2(z)$ *which makes the system (5.6) asymptotically stable. Then the formulas (5.3), (5.4) and (5.5) along with this* $b_2(z)$ *provide a compensator which solves* **IODISFBK**.

Proof: The set of states reachable from 0 for the closed loop system with the given choice of compensator is a subset of the diagonal $\mathcal{D}$ since $\mathcal{D}$ is invariant for the closed loop system (5.6).

Apply Proposition 1.1 with $\varepsilon(x,z) = \eta(x)$ to the system defined by restricting the closed loop system to the diagonal $\mathcal{D}$. Then

$$\mathcal{H}|_{\mathcal{D}} = H^{*W}_{a^*(x),b_1(x),b_2(x),c^*(x)}(x,x) = IAX(x) \le 0,$$

i.e. the ε-**DISFBK** inequality holds on the diagonal. It follows that the **IODISFBK** problem is solved with the given compensator, since the closed loop system is asymptotically stable by assumption. ∎

The main remaining question is that of choosing b_2 in the compensator. According to Theorem 5.2 the **IODISFBK** problem can be solved in two separated steps: 1) solve $IAX(x) \le 0$ for a nonnegative η, where $X = \nabla\eta(x)$; 2) choose $b_2(z)$ so that the closed loop system is asymptotically stable. For the second step we consider the following system:

$$\begin{aligned} \dot{x} &= F_1(x,z) \\ \dot{z} &= F_1(z,z) - v(C_2(z) - C_2(x)) \end{aligned} \tag{5.7}$$

with z as output and v as input, where

$$F_1(x,z) := A(x) - B_2(x)e_1(z)^{-1}[B_2(z)^T X(z) + D_{12}(z)^T C_1(z)].$$

Comparing (5.7) to (5.6) we can easily see that choosing $b_2(z)$ so that the system (5.6) (with $W_1 = 0$) is asymptotically stable is a standard static output feedback stabilization problem for the system (5.7). We refer the interested reader to Helmke and Anderson 1992 on this subject which is still an open problem. In summary we have the following:

Corollary 5.3. *The* **IODISFBK** *problem is solvable with the choice of (5.3-5.5) if and only if*

(1) there exists $\eta \ge 0$ such that $\eta(0) = 0$ and $X(x) = \nabla\eta(x)$ solves $IAX(x) \le 0$, and

(2) the system (5.7) is stabilizable via static output feedback.

In the above corollary we restricted ourselves to compensators which have same dimension as the original plant state space. It is not clear whether there will be any advantage to increasing the compensator order as far as dissipation is concerned. But it is very possible that we may have better results on stability with compensators of higher order. A natural configuration is described in the following proposition which includes Corollary 5.3 as a special case (i.e., when $dim(\xi) = 0$).

Proposition 5.4. *The* **IODISFBK** *problem is solvable with the choice of (5.3-5.5) if*

(1) there exists $\eta \geq 0$ *such that* $\eta(0) = 0$ *and* $X(x) = \nabla\eta(x)$ *solves* $IAX(x) \leq 0$*;*

(2) the system (5.7) is stabilizable via dynamic output feedback

$$\dot{\xi} = d(\xi, z), \quad v = h(\xi, z).$$

Proof: We use the same z equation except replacing $b_2(z)$ by $b_2(\xi, z)$ where ξ satisfies

$$\dot{\xi} = d(\xi, z).$$

First observe that for the corresponding closed loop system $\mathcal{D}$ is still an invariant submanifold. To prove this we set $x = z$ in the differential equations of the closed loop system, then it follows from (5.7) that $\dot{x} = \dot{z}$.

If (1) holds then we have an energy function $\varepsilon(x, z) = \eta(x)$. It is easy to show that we have

$$H^{*W}|_{\mathcal{D}} = IAX(x)$$

which is independent of the choice of the vector field $d(\xi, z)$. Starting from 0, the closed loop system will stay on the invariant submanifold $\mathcal{D}$. Therefore the dissipation inequality always holds due to (1).

Choosing $b_2(\xi, z) = h(\xi, z)$, then the stability of the closed loop system follows from (2). ■

Our main question has been reduced to an output feedback stabilization problem, but it is a classical difficult problem. The reader is welcome to try his hand applying the classical literature to this problem. However we now turn to using the special structure of our system (5.6) to treat the stability problem. We describe several approaches in the next a few sections.

b. Global output feedback stabilization

The analysis in this section is based on using a Lyapunov function of a special form. This of course is overly restrictive but indeed it gives informative results. We assume the closed loop system has a Lyapunov function consisting of two parts, a quadratic function of $(x-z)$ and a nonlinear function of x and analyze the consequences. The resulting theorem incidentally gives conditions which are necessary and sufficient for linear systems.

Proposition 5.6. *Suppose* $C_2(x) = C_2 x$, $IAX(x) < 0$ *has a proper positive definite solution* $\eta(x)$ *with* $X(x) = \nabla\eta(x)$ *(proper means that for each* $c > 0$ *the set* $\{x \in R^n | 0 \leq \eta(x) \leq c\}$ *is compact), and there exists a positive definite matrix* P *such that for some* $\rho(z) \geq 0$

$$2(x-z)^T P\Big\{A(x) - A(z) - [B_2(x) - B_2(z)]e_1(z)^{-1}\Big[B_2^T(z)X(z) + D_{12}(z)^T C_1(z)\Big]\Big\} +$$

$$X(x)^T\Big\{A(x) - B_2(x)e_1(z)^{-1}\Big[B_2^T(z)X(z) + D_{12}(z)^T C_1(z)\Big]\Big\} - \rho(z)(x-z)^T C_2^T C_2(x-z) < 0$$

Then the **IODISFBK** *problem has a global solution. In this case one can choose* $b_2(z) = \rho(z)P^{-1}C_2^T$.

Proof: Choose the following candidate for a Lyapunov function: $V(x,z) = \eta(x) + (x-z)^T P(x-z)$. Clearly $V(x,z) > 0$. From (5.6) (with $W_1 = 0$), we have

$$\dot{V}(x,z) = X(x)^T\dot{x} + 2(x-z)P(\dot{x} - \dot{z})$$

$$= X(x)^T\Big\{A(x) - B_2(x)e_1(z)^{-1}\Big[B_2(z)^T X(z) + D_{12}(z)C_1(z)\Big]\Big\} +$$

$$2(x-z)^T P\Big\{A(x) - A(z) + [B_2(z) - B_2(x)]e_1(z)^{-1}\Big[B_2(z)^T X(z) + D_{12}(z)^T C_1(z)\Big] - b_2(z)C_2(x-z)\Big\}$$

Let $b_2(z) = \rho(z)P^{-1}C_2^T$. The assumptions imply that $\dot{V}(x,z) < 0$. By Theorem 5.2 we have a solution to the **IODISFBK** problem. ■

c. Local output feedback stabilization

When we are only interested in the performance of a nonlinear system in a sufficiently small neighborhood of 0, then we are able to obtain very simple conditions. A somewhat trivial argument for local result is: using Taylor expansion for the Hamiltonian, if the second order terms are negative definite then we can use the first order approximation and linear design to achieve the required performance level locally around 0. This argument would fail if we have a critical case, e.g., the second order terms in the Hamiltonian is negative semi-definite. In such case the higher order terms determine the local property of H. In this subsection we try to develop some local results which can handle some critical cases. More specifically, we do not require $IAX(x)$ ($\eta(x)$) to be bounded by negative (positive) definite quadratic functions.

Theorem 5.7. *The* **IODISFBK** *problem is solvable with the choice (5.3-5.5) of compensator if*
(1) the trajectories of the closed loop system stay in a sufficiently small neighborhood U *of the origin;*
(2) (A_0, C_{20}) *is detectable where* $A_0 = \frac{\partial A(x)}{\partial x}|_{x=0}, C_{20} = \frac{\partial C_2(x)}{\partial x}|_{x=0}$ *and;*
(3) there exists a positive definite $\eta(x)$ *such that* $X(x) = \nabla\eta(x)$ *solves* $IAX(x) < 0$ *for all* x *such that* $(x,x) \in U$.

Proof: The proof is based on a Lyapunov function of the form

$$V(x) = (x-z)^T P(x-z) + \eta(x)$$

and it is not surprising. Thus we omit it. ■

Also one can prove:

Proposition 5.8. *For linear systems the* **IODISFBK** *problem is solvable with* a, b_1, c *given by (5.3-5.5) if and only if* $L_X < 0$ *has a positive definite solution* X *and* (A, C_2) *is detectable.*

Remark For linear systems we can prove that the conditions in Proposition 5.8 imply those in Proposition 5.6. Thus Proposition 5.6 gives necessary and sufficient conditions for the linear H^∞ control problem with a, b_1, c given by (5.3-5.5).

d. ε-DISFBK in a neighborhood of $\mathcal{D}$

According to the results in Section 5a, it is enough in theory to solve $IAX(x) \leq 0$ on the diagonal and choose b_2 to make the closed loop system stable, since the diagonal $\mathcal{D}$ is the set of reachable states of the closed loop system. In practice however we wish to achieve the performance goal (**IODISFBK**) in a more robust manner since we may not know the initial condition of the system precisely and the system may be subject to perturbations. Localizing around the diagonal is very natural for this problem, so even though our results are not close to implementation we include them here. Again we consider the system (5.1) without the disturbance signal W_2.

Proposition 5.9. *Assume that* $D_{21}(x) = D_{21}, D_{12}(x) = D_{12}, B_1(x) = B_1, B_2(x) = B_2$ *are constant functions,* $D_{12}^T D_{12} = I$, $D_{21} D_{21} = I$, $D_{12}^T C_1(x) = 0$, $B_1 D_{21}^T = 0$, *and all functions defining the plant are smooth. Define*

$$IAV(x) = \mathcal{V}(x)^T \mathcal{A}(x) + \mathcal{A}(x)^T \mathcal{V}(x) + 2 D_x X(x)^T B_2 B_2^T D_x X(x) \tag{5.8}$$

where

$$\mathcal{V}(x) = \begin{bmatrix} 2V(x) \\ \\ D_x V(x) \end{bmatrix} \quad \text{and} \quad \mathcal{A}(x) = \begin{bmatrix} D_x A(x) \\ \\ A(x) + (B_1 B_1^T - B_2 B_2^T) X(x) \end{bmatrix}$$

and the multiplication $D_x V(x)^T [A(x) + (B_1 B_1^T - B_2 B_2^T) X(x)] := V'(x)$ *means that the elements of the matrix* $V'(x)$ *satisfy* $(V'(x))_{i,j} = (\nabla V_{i,j}(x))^T [A(x) + (B_1 B_1^T - B_2 B_2^T) X(x)]$.

If there is a smooth solution $X(x)$ *to* $IAX(x) \leq 0$ *which is the gradient of a nonnegative* *function* $\eta(x)$, $\eta(0) = 0$ *and matrix* $V(x) > 0$ *and function* $r(x) > 0$ *satisfying*

$$IAV(x) - r(x)D_xC_2(x)^T D_xC_2(x) < 0 \tag{5.9}$$

then there exists $\varepsilon(x,z)$ *satisfying* $\varepsilon \geq 0$ *in a neighborhood* Ω *of the diagonal* $\mathcal{D}$, $\varepsilon(0,0) = 0$, *and satisfying* $Z_\varepsilon = \mathcal{D}$ *so that the compensator defined by (5.3-5.5) with* b_2 *given by*

$$V(x)b_2(x) = [D_xC_2(x)]^T r(x) \tag{5.10}$$

and ε *solve* ε-**DISFBK** *near the diagonal* $\mathcal{D}$. *If* $\eta(x)$ *is a proper function, then trajectories of the closed loop system (5.6) which remain in* Ω *for* $t \geq 0$ *are bounded. If* $\eta(x)$ *is proper with* $\eta(x) > 0$ *and* $IAX(x) < 0$ *for* $x \neq 0$, *then trajectories of the closed loop system (5.6) which remain in* Ω *for* $t \geq 0$ *satisfy* $(x(t), z(t)) \to 0$ *as* $t \to \infty$.

Conversely, If $\varepsilon(x,z)$ *solves* ε-**DISFBK** *such that* $Z_\varepsilon \supseteq \mathcal{D}$, $D_z(\nabla_z\varepsilon(x,z))\big|_{z=x}$ *has full rank, and* $IAX(x) \equiv 0$, *then there exist* $V(x) \geq 0$ *and* $b_2(x)$ *such that*

$$IAV(x) - 2V(x)b_2(x)D_xC_2(x) - D_xX(x)^T B_1B_1^T D_xX(x) \leq 0.$$

Proof: (1) Take $a(z), b_1(z), c(z)$ as (5.3-5.5), and define

$$\varepsilon(x,z) = 2\eta(x) + (z-x)^T V(x)(z-x)$$

where $X(x) = \nabla\eta(x) = \frac{1}{2}\nabla_x\varepsilon(x,x)$ solves $IAX \leq 0$ and $\eta \geq \eta(0)$ (easily normalized to $\eta(0) = 0$) is guaranteed by hypothesis and $V(x) > 0$ is the matrix satisfying (5.9).

With this choice of compensator and $\varepsilon(x,z)$,

$$\nabla_x\varepsilon(x,z) = 2X(x) - 2V(x)(z-x) + (z-x)^T D_xV(x)(z-x)$$

$$\nabla_z\varepsilon(x,z) = 2V(x)(z-x), \qquad \text{and}$$

$$H^{*W}(x,z) = IAX(x) + (z-x)^T\Big[IAV(x) - r(x)D_xC_2(x)^TD_xC_2(x)\Big](z-x) + O(\|z-x\|^3)$$

Thus the closed loop system with the proposed compensator solves ε-**DISFBK** near the diagonal $\mathcal{D}$. Define

$$\Omega = \{(x,z) : \varepsilon(x,z) > 0, H^{*W}(x,z) < 0\} \bigcup \mathcal{D}.$$

Then Ω is open by construction, and a proof similar to Ball *et al* 1993 Theorem 5.1 will show that trajectories remaining in Ω for $t \geq 0$ are bounded. Proof of $(x(t), z(t)) \to 0$ as $t \to \infty$ is a modification of LaSalle and Lefschetz 1961 Theorem IX.

(2) By Theorem 3.1, if $IAX(x) \equiv 0$, then $c(z)$ and $a(z)$ are given by (5.3) and (5.4) respectively. Since $Z_\varepsilon \supseteq \mathcal{D}$, ε is given by

$$\varepsilon(x,z) = 2\eta(x) + (z-x)^TV(x)(z-x) + O(\|z-x\|^3).$$

We must have

$$\begin{aligned}&0 \geq H^{*W}_{a^*(z),b_1(z),b_2(z),c^*(z)}(x,z) \geq \min_{b_1} H^{*W}_{a^*(z),b_1,b_2(z),c^*(z)}(x,z)\\ &=(z-x)^T[IAV(x) - 2V(x)b_2(x)D_xC_2(x) - D_xX(x)^TB_1B_1^TD_xX(x)](z-x) + O(\|z-x\|^3)\end{aligned}$$

where the minimizing $b_1(z)$ is given by

$$b_1(z)^T\nabla_z\varepsilon(x,z) = -B_1^T\nabla_x\varepsilon(x,z) + 2B_1^TX(z) - D_{21}^Tb_2(z)^T\nabla_z\varepsilon(x,z)$$

Therefore

$$IAV(x) - 2V(x)b_2(x)D_xC_2(x) - D_xX(x)^TB_1B_1^TD_xX(x) \leq 0.$$

■

Note the unconventional notation in (5.8) where the upper and lower blocks in $\mathcal{V}$ are a matrix and a third order tensor, respectively, while the upper and lower blocks in $\mathcal{A}$ are a matrix and a vector.

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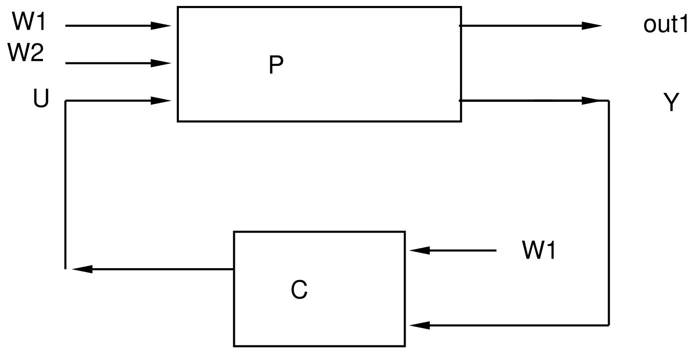

Figure 1.

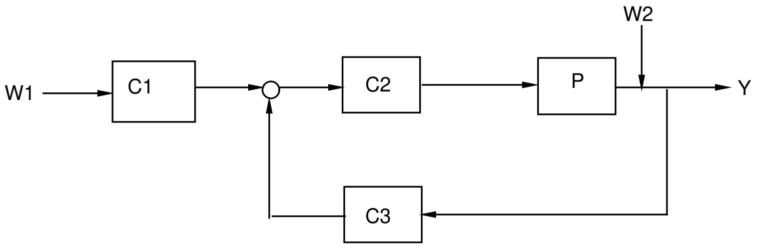

Figure 2.