Some Systems Theorems Arising From The Bieberbach Conjecture

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The purpose of this note is to describe the system theoretic component of the proof of the Bieberbach conjecture. We were quite surprised to find strong connections to modern robust control theory. Much of the mathematical content of this note comes directly from the paper of Vasyunin and Nikolskii [**VN**] which in turn is heavily dependent on the original proof by de Branges [**dB**] and and Theorem 94 of the unpublished manuscript [**dB4**]. Also the reader is refered to a system theoretic approach [**dB2**]. One of the contributions of this paper is to identify key constraints and estimates in [**VN**] as very natural engineering systems constraints. We were extremely surprised by the extent this was possible after slightly modifying the class of systems treated by [**VN**]. Another contribution is to extend the generality in [**VN**] from systems with no output and with invertible input operators to conventional $[A, B, C]$ systems. Our objective in the paper is not to actually give a full proof of the Bieberbach conjecture but to extract the systems ideas which might be of potential use to system theorists and mathematicians. Since our goal is to make a paper easily readable to system theorists, we operate at a different level of generality than [**VN**]. While algebraically our results are more general than [**VN**], we do not have time varying

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input and output spaces, unbounded operators, or worry with technical issues in Hilbert space.

The [**VN**] proof might be thought of in four parts, corresponding to the four parts of this paper. Conceptually, the first is general systems theory and actually contains a refinement of the classical Bounded Real Lemma, BRL, which is new even in finite dimensions. The second part (after a modification of [**VN**]) is a BRL for a convex family SYS^K of systems; indeed it is a robustness result of currently fashionable type. The third is to develop a test to determine if there is a uniform bound on the input output operator in SYS^K. To actually put teeth in the general systems theorem requires a strong assumption. In this case it is roughly: the extreme points of the convex set of systems are systems all of which have the same frequency response function. Under a bit stronger assumption the technique [**VN**] call chronological averaging applies to reduce the uniform bound computation for all of SYS^K to solving a Riccati equation associated with just one system. The last part pertains only to "Löwner systems" and while very specialized gives an impressive example of how these general theorems apply. Here we refer the reader mostly to [**VN**] for an excellent rigorous proof, but sketch a formal proof and give enough details to verify how the modifications we made in parts II and III (in order to obtain conventional systems theorems) blends quickly into the [**VN**] line of proof.

PART I. DISSIPATIVE SYSTEMS

We begin with a time varying system

$$\frac{dx}{dt}(t) = A(t)x(t) + B(t)u(t) \tag{I.1}$$

$$y(t) = C(t)x(t)$$

and later extend our study to a family of systems ranging through $A(t)$ lying in a given convex set K. We ultimately seek energy-type estimates which are valid for all systems with $A(t)$ being in K. Thus we shall be interested in a robustness type of result.

The input, output, and state spaces (U, Y, and X respectively) are complex Hilbert spaces and we impose various norms on them denoted by

(on inputs) $$\|u\|_{\Gamma(t)} = \|\Gamma(t)^{-1}u\|$$

(on outputs) $$\|y\|_{\beta(t)} = \|\beta(t)^{-1}y\|$$

(on states) $$\|x\|_t^2 = (\sigma(t)x, x)$$

where $\sigma(t)$ is a positive definite operator on the state space and $\Gamma(t)$, $\beta(t)$ are bounded invertible operators. Also as functions of t we assume they are continuous. The use of subscripts on $\| \ \|$ is consistent with [**VN**]. The operators $A(t)$, $B(t)$, $C(t)$ act on the usual state, input, and output spaces and are everywhere defined and bounded on them. The reader uncomfortable with infinite dimensional spaces can think of these as finite dimensional since the points we emphasize in this article are also new in finite dimensions. To prove the Bieberbach conjecture one actually needs to permit greater generality, for instance A must be allowed to be a dissipative differential operator which is unbounded.

Actually we could take the input, output, and state spaces not to be fixed but to be sets which depend on time. This level of generality is required to prove the Bieberbach conjecture (see [**VN**]). Treating the time varying weights $\beta(t)$, $\Gamma(t)$, $\sigma(t)$ actually contains much of the formal idea of going to time varying spaces, and so we do not add the considerable notation required for time varying spaces.

We shall be concerned with dissipativity of the system $[A, B, C]$. We say it is IO *dissipative* provided

$$\int_0^r \|y(s)\|^2_{\beta(s)}\, ds \le \int_0^r \|u(s)\|^2_{\Gamma(s)}\, ds \quad \forall\, r$$

for all inputs u and resulting outputs provided the system is started in state $x(0) = 0$.

The solution x of the system can be obtained in an integral form by introducing *evolution* operators T_{tr}, $0 \le r \le s \le t \le b$ which satisfy

$$\frac{\partial T_{tr}}{\partial r} = -T_{tr} A(r) \qquad \frac{\partial T_{tr}}{\partial t} = A(t) T_{tr} \ .$$

The solution is

$$x(t) = T_{tr} x(r) + \int_r^t T_{ts} B(s) u(s)\, ds \ .$$

The *controllability* operator $\mathcal{C}_r^t$ is defined on input functions by

$$[\mathcal{C}_r^t u](t) = \int_r^t T_{ts} B(s) u(s)\; ds \ .$$

Its value is the state $x(t)$ at time t attained by the system started at $x(r) = 0$ and driven by the function u on $[r, t]$. For the fixed input u, the *observability* operator maps each state x_0 to the output function on $[0, \infty]$ which results from initializing the system in state x_0.

A standard result (referenced below) says:

THEOREM I.1. *Suppose the Riccati inequality*

$$-\frac{d\sigma}{dt}(t) - \sigma(t)B(t)\Gamma(t)\Gamma^*(t)B(t)^*\sigma(t) - \sigma(t)A(t) \\ - A(t)^*\sigma(t) - C^*(t)\beta(t)^{*-1}\beta(t)^{-1}C(t) \geq 0 \tag{I.2}$$

has a solution $\sigma(t)$ for all $t \geq 0$ which is positive definite, bounded, and invertible on X. Then the system (I.1) is IO *dissipative. Also under technical assumptions on A, B, and C (irrelevant to this paper) there is a $\sigma(t)$ which satisfies (I.2) with equality in a weakened form.*

Addendum. *Moreover, the controllability operator $\mathcal{C}_r^t$ is a contraction into the σ norm (energy norm) on the state space, that is*

$$(\sigma(t)\mathcal{C}_r^t u, \mathcal{C}_r^t u) \leq \int_r^t \|u(s)\|_{\Gamma(s)}^2 \, ds \tag{I.3}$$

for input functions u, and provided $x(r) = 0$.

A similar inequality

$$\int_r^t \|C(s)x(s)\|_{\beta(s)}^2 \, ds \leq (\sigma(r)x(r), x(r)) - (\sigma(t)x(t), x(t)) \tag{I.4}$$

holds for the observability operator where we set input $u \equiv 0$, and initialize at state $x(r)$.

Remarks. (1) For the time invariant case Theorem I.1 is the standard Bounded Real Lemma due to Brian Anderson. This comes from the electric circuits culture. Control theory also produced a similar result which is due to Kalman- Yacubovich- Potopov with extremely notable work by Jan Willems [**W**] and by Hill-Moylan [**HM**]. Also this is the subject of the Pontrayagin maximal principle. There are many references (see [**W**] for a list). One which operates at a high level of generality is Theorem 3, §5.5.2 [**BO**]. One of the most succinct summaries of time invariant results is Theorem 2 [**vS**]. The [**vS**] proof is for nonlinear systems and can easily be extended to time varying systems.

(2) What is new in this part of the paper to systems theory is just the addendum. It is well known that the "Hankel operator" for an IO dissipative system equals the observability operator times the controllability operator. Further it is known that the Hankel Operator is a contractive operator. What we establish here is a more refined structure: in the $\sigma(t)$ metric both the observability and controllability operators are contractions.

(3) In applications to the Bieberbach conjecture the addendum shall be specialized to an equation discovered by Löwner describing all conformal maps on the unit disk. The inequality (I.3) is the key estimate needed to solve the conjecture.

While the proof of the addendum (at the level of generality presented here) is easy, the fact appears to be new to systems theory. In order to prove the addendum one needs to develop most of the proof of Theorem I.1. Since references have been given where complete treatments can be found we shall keep the level of proof high and try to sketch the idea behind Theorem I.1 itself.

The inequality (I.2) is the "linear system" special case of the Hamilton-Jacobi-Bellman-Issacs equation (inequality version) and itself, traces back to Kalman and Bucy. Inequality (I.5) below is how the HJBI is usually written. Under reasonable hypotheses if a solution to inequality (I.2) exists, then a solution giving equality also exists. We wish to show that

(I.2) guarantees that the system is dissipative. The intuitive idea involves a time varying potential energy e of the state $x \in X$ which for a dissipative system satisfies

$$e(t_1, x(t_1)) + \int_{t_1}^{t_2} \|u(s)\|^2_{\Gamma(s)} - \|y(s)\|^2_{\beta(s)}\, ds \quad \geq \quad e(t_2, x(t_2))$$

for any times $t_1 < t_2$. That is,

$$e\ (\text{present state}) \ + \ \text{energy in} \ - \ \text{energy out} \ \geq e\ (\text{future state})\ .$$

The function e is called a *storage function* by circuit and control theorists and a *value function* in game theory. Now convert this to infinitesimal form by writing

$$\frac{e(t_1, x(t_1)) - e(t_2, x(t_2))}{t_1 - t_2} + \frac{1}{t_1 - t_2} \int_{t_1}^{t_2} \|u(t)\|^2_{\Gamma(t)} - \|y(t)\|^2_{\beta(t)}\, dt \leq 0$$

and taking the limit as $t_2 \to t_1$. This gives

$$\text{(I.5)} \qquad -\frac{de}{dt}(t_1, x(t_1)) + \|u(t_1)\|^2_{\Gamma(t_1)} - \|y(t_1)\|^2_{\beta(t_1)} \geq 0\ .$$

That (I.2) has a solution implies (I.2) with equality has a solution follows from analyzing an integral representation for e, such as

$$e(t_1, x(t_1)) = \inf_u \lim_{t_2 \to \infty} \int_{t_1}^{t_2} \|u(s)\|^2_{\Gamma(s)} - \|y(s)\|^2_{\beta(s)}\, ds\ .$$

This we shall not do.

Since our systems are linear we may take e to be quadratic and indeed of the form $e(t,x) = (\sigma(t)x, x)$. Direct calculation shows

$$-\frac{d}{dt}\,(\sigma(t)\,x, x) = \left(-\frac{d\sigma(t)}{dt}\,x, x\right) - 2\mathrm{Re}\,(\sigma(t)x, Ax + Bu).$$

So (I.5) is the same as

$$\text{(I.6)} \qquad -\left(\frac{d\sigma(t)}{dt}\,x, x\right) + \mathcal{H}(t,x,u) \geq 0$$

for all $x \in X$, t real, and $u \in U$. Here $\mathcal{H}$ is given by

$$\text{(I.7)} \qquad \mathcal{H}(t,x,u) = \|u\|^2_{\Gamma(t)} - \|y\|^2_{\beta(t)} - 2\mathrm{Re}\,(\sigma(t)x, A(t)x + B(t)u)$$

and is called the Hamiltonian for the system.

Proof of Theorem I.1. Clearly such a Hamiltonian as in (I.7) can be associated to a given system and a given $\sigma(t)$. The u which minimizes (I.7) is gotten by setting the differential of $\mathcal{H}$ in u to zero and solving for u. It is

$$\text{(I.8)} \qquad \tilde{u}(t) = \Gamma(t)\Gamma(t)^* B(t)^* \sigma(t) x \; .$$

Inserting it in $\mathcal{H}$ produces essentially the left side of (I.2) without the term $\left(-\frac{d\sigma(t)}{dt}\,x, x\right)$. Thus (I.2) simply says that

$$\text{(I.9)} \qquad -\left(\frac{d\sigma(t)}{dt}\,x, x\right) + \min_u \; \mathcal{H}(t,x,u) \geq 0 \; ,$$

for all $x \in X$ and t real. This in turn is precisely the Hamilton-Jacobi-Bellman-Issacs inequality guaranteeing IO dissipativity [**BO**, Ch.5]. Here the assumption of boundedness of the $A(t), B(t)$ is necessary to guarantee obtaining the inequality for all x. If they are unbounded with time varying domains as in the Bieberbach problem there are considerable difficulties. Note that if (I.2) holds with equality then (I.9) becomes an equality.

Proof of Addendum. To see that the Riccati inequality (I.2) implies the controllability inequality (I.3), rearrange (I.9) and drop the term $\|y\|^2$ (which only strengthens the inequality) to get

$$\frac{d}{ds}\,(\sigma(s)x(s), x(s)) \le \|u(s)\|^2_{\Gamma(s)}$$

for all input functions u. Integrate this to obtain

$$(\sigma(t)x(t), x(t)) \le \int_r^t \|u(s)\|^2_{\Gamma(s)}\, ds$$

for all u provided $x(r) = 0$; which is just a restatement of (I.3).

To show (I.4), the observability inequality, observe that in infinitesimal form (I.4) reads

$$\|C(r)x(r)\|^2_{\beta(r)} \le -\frac{d}{dr}\,(\sigma(r)x(r), x(r))$$

which is the same as (I.5), and hence (I.6), with $u = 0$. Thus the observability inequality also follows from (I.9).

Remark. When $C^*\beta^{*-1}\beta^{-1}C = 0$, as is the case in the Bieberbach problem, the Riccati inequality (I.2) has no zero order term in σ. Thus the usual Legendre transform which amounts to multiplying by σ^{-1} on both sides converts (I.2) into a linear inequality

$$\frac{d\Sigma}{dt} - B\Gamma\Gamma^* B^* - \Sigma A^* - A\Sigma \ge 0$$

in $\Sigma = \sigma^{-1}$. In [**VN**] they obtain a linear equation (I104) by a similar but less standard transformation.

A practical and elegant reformulation of the Riccati Inequality (I.2) can be given in terms of *Hamiltonian Systems.* To describe these the dependence of $\mathcal{H}(t, x, u)$ on σ is made explicit by the introduction of the *costate vector* $p=\sigma(t)x$ (beware that many texts take $p=2\sigma(t)x$), and $\mathcal{H}$ is minimized over u. Thus, define

$$H(t,x,p)=\min_u\ \mathcal{H}(t,x,u)\geq 0$$

where p replaces $\sigma(t)x$ in the definition (I.7) of $\mathcal{H}$. The Hamiltonian system associated with our dissipativity problem whose solution is equivalent to solving (I.2) is

$$\frac{dx(t)}{dt}=\frac{\partial H(t,x,p)}{\partial p} \qquad \frac{dp(t)}{dt}=-\frac{\partial H(t,x,p)}{\partial x}\ .$$

For our system (I.1) one computes the u minimizer as in (I.8) to be $\tilde{u}=\Gamma(t)\Gamma(t)^*B(t)^*p$, and substitutes this into $\mathcal{H}$ to get H equal to

$$(-B(t)\Gamma(t)\Gamma^*(t)B(t)^*p,p)-(A(t)x,p)-(A(t)^*p,x)-\big(C(t)^*\beta(t)^{*-1}\beta(t)^{-1}C(t)x,x\big)\ .$$

Thus our associated Hamiltonian system is †

$$\text{(I.10x)} \qquad \frac{dx}{dt} = Ax + B\Gamma\Gamma^* B^* p$$

$$\text{(I.10p)} \qquad \frac{dp}{dt} + A^* p + C^* \beta^{*-1} \beta^{-1} C x = 0 \; .$$

We shall return to these equations in part III.

PART II. FAMILIES OF SYSTEMS

Now we study not just one system but the dissipativity of every system in a convex family of systems. We investigate a common type of robustness question in system theory; namely does every system in a given family meet the performance specifications. This section is a variation of paragraphs (I120) and (I130) of [**VN**], and presents an alteration of the technique they call chronological averaging.

† In order to be self contained we mention that (I.10x) can be identified as the closed loop system version of (I.1) with feedback $\tilde{u} = \Gamma\Gamma^* B^* \sigma x$ as state dynamics. To get a little more generality, observe that in this closed loop system $\sigma \frac{dx}{dt} + \frac{d\sigma}{dt} \; x = \sigma A x + \left(\frac{d\sigma}{dt} \; + \sigma B \Gamma\Gamma^* B^* \sigma \right) x$ which by the Riccati inequality (I.2) gives

$$\begin{aligned} \sigma \frac{dx}{dt} + \frac{d\sigma}{dt} x &= \sigma A x - \{ \sigma A + A^* \sigma + C^* \beta^{*-1} \beta^{-1} C \} x - \mathcal{P} x \\ &= -A^* \sigma x - C^* \beta^{*-1} \beta C x - \mathcal{P} x \end{aligned}$$

where $\mathcal{P}$ is a non-negative definite operator. From this we obtain

$$\left(\frac{d\sigma x}{dt} + A^* \sigma x + C^* \beta^{*-1} \beta^{-1} C x, x \right) \leq 0 \quad \forall x.$$

Since Hamiltonians are not mentioned in [**VN**] (I.10x) and (I.10p) are not identified as the usual Hamiltonian system and different notation is used. They study the case where $\mathcal{P} = 0$ and $C = 0$. Then (I.10p) becomes

$$\text{(I.10b)} \qquad \frac{d\sigma x}{dt} + A^* \sigma x = 0$$

which is basically a generalized version of the de Branges equation [**dB**, p.141].

Let K be a convex set of time invariant systems $[A, B, C]$, and assume the extreme points E of K form a compact metric space. Here a convex combination is in the sense

$$[A, B, C] = [rA_1 + (1-r)A_2, rB_1 + (1-r)B_2, rC_1 + (1-r)C_2]$$

where r is a positive number less than or equal to 1. Of course the IO-map for $[A, B, C]$ is typically not a convex combination of the IO-maps for $[A_1, B_1, C_1]$ and $[A_2, B_2, C_2]$. Since the extreme point set E is compact, the Choquet Theorem applies and says that any $[A, B, C]$ in K can be represented as

$$A = \int_E A_\zeta \; d\mu(\zeta) \; , \quad B = \int_E B_\zeta \; d\mu(\zeta) \; , \text{ and } \quad C = \int_E C_\zeta \; d\mu(\zeta)$$

where $[A_\zeta, B_\zeta, C_\zeta] \in E$, and where μ is a positive measure on E of total mass equal to 1. Often we abbreviate and instead of referring to $[A_\zeta, B_\zeta, C_\zeta] \in E$ we use the notation $\zeta \in E$.

Our main interest is in the class of time varying systems, we use the notation SYS^K to represent all systems (I.1) with $[A(t), B(t), C(t)] \in K$ for all t (technically speaking for a.e. t). The next theorem estimates behavior of any system in the class SYS^K in terms of systems in E (even though E is just a subset of the extreme systems of SYS^K). This of course is very powerful since systems in E are time invariant and, as with the Bieberbach problem, one can often explicitly solve them. In this Theorem, and throughout the rest of the paper, the term *storage function* shall refer to a function σ satisfying the Riccati inequality (I.2).

THEOREM II.1. *If there is a single storage function σ_* which satisfies (I.2) for all systems in E, then σ_* also satisfies (I.2) for any system in SYS^K. Consequently if*

$\sigma_* \geq 0$, then all systems in SYS^K are dissipative and the controllability and observability operator for such systems are contractions with respect to σ_* as in (I.3) and (I.4). Here σ_* is a time varying storage function even though the systems in E are time invariant.

Proof. The trick of de Branges applies to special systems but for the systems here we need more generality and we use the recent engineering technique of converting a Riccati inequality to a linear matrix inequality (LMI). This method has been championed by Steve Boyd (see [**B**]). It is straightforward using Schur complements to check that inequality (I.2) is equivalent to the statement that the block matrix

$$\mathcal{M}(A,B,C,\sigma(t)) = \begin{pmatrix} \frac{d\sigma}{dt} + A^*(t)\sigma(t) + \sigma(t)A(t) & \sigma(t)B(t)\Gamma(t) & C^*(t)\beta(t)^{*-1} \\ \\ \Gamma(t)^* B^*(t)\sigma(t) & -I & 0 \\ \\ \beta(t)^{-1}C(t) & 0 & -I \end{pmatrix}$$

is negative semidefinite. The Algebraic Riccati version of this appears in Lu and Doyle [**LD**]. The point is that $\mathcal{M}$ is (real affine) linear in A, B, C, and σ, thus

$$\mathcal{M}(A(t), B(t), C(t), \sigma(t)) = \int_E \mathcal{M}(A_\zeta, B_\zeta, C_\zeta, \sigma(t))\, d\mu_t(\zeta)\,.$$

If $\sigma(t) = \sigma_*$, then the integrand on the right is negative semidefinite for all ζ, and so the integral is negative semidefinite. Thus (I.2) holds for each t, which proves the theorem.

This is rather straightforward though recent system theory. The remaining difficulty is in finding one σ_* which is a storage function for all systems in E. This requires further special structure of the system. A key assumption, motivated by the situation in the

Bieberbach problem, is that of *equivalence of extreme points* (or EEP for short) by which we mean:

All systems $[A_\zeta, B_\zeta, C_\zeta] \in E$ are unitarily equivalent: For a fixed ζ_0, there is a unitary operator $v(\zeta)$ on the state space so that

$$(\text{II.1}) \qquad A_\zeta = v(\zeta)A_{\zeta_0}v(\zeta)^{-1}\ , \quad B_\zeta = v(\zeta)B_{\zeta_0}\ , \quad C_\zeta = C_{\zeta_0}v(\zeta)^{-1}.$$

Moreover, E and $\{v(\zeta) : \zeta \in E\}$ are assumed to be compact groups.

This of course is a very strong assumption, in [**VN**] such an assumption is used to perform an averaging technique they call *chronological averaging.*

One immediate consequence of EEP is that all systems in E have the same input output operator, since they have the same transfer function $C_\zeta(sI - A_\zeta)^{-1}B_\zeta$. Conversely, any controllable and observable system with the same input output operator as $[A_\zeta, B_\zeta, C_\zeta] \in E$ has the form EEP with a possibly non-unitary $v(\zeta)$. Another property of E, under the assumption of EEP, pertains to Theorem I.1, the Bounded Real Lemma. This is described in the next Lemma, which shall be used in part III. Note that in this Lemma and its proof the σ's are functions of t but we do not indicate this each time we write them.

LEMMA II.2. *Assume EEP.*

(a) If σ_* satisfies (I.2) for a single system in E and commutes with $v(\zeta)$ for all ζ in E, then σ_* is a storage function for all systems in E and consequently, by Theorem II.1, for all systems in SYS^K.

*(b) (equation C102 [**VN**]) If σ_0 satisfies (I.2) for all systems in E, then there exists a σ_* commuting with all $v(\zeta)$ and satisfying (I.2) for all systems in E, and hence in SYS^K. Further, if $\sigma_0 \geq 0$ then we may take $\sigma_* \geq 0$.*

Proof. (a) Assume σ_* satisfies (I.2) for the system $[A_{\zeta_0}, B_{\zeta_0}, C_{\zeta_0}] \in E$ (that is with $A = A_{\zeta_0}$, $B = B_{\zeta_0}$, and $C = C_{\zeta_0}$ in (I.2)), then

$$\sigma_\zeta = v(\zeta)\sigma_* v(\zeta)^{-1}$$

satisfies (I.2) for all $[A_\zeta, B_\zeta, C_\zeta] \in E$. This one reads off from (I.2) and (II.1). If in addition σ_* commutes with $v(\zeta)$, then $\sigma_* = \sigma_\zeta$ satisfies (I.2) for all systems in E as required.

(b) This part uses the averaging technique of [**VN**]. Let $d\zeta$ denote Haar measure (invariant measure) on the group E. It always exists on a compact group. Assume σ_0 satisfies (I.2) for all systems in E, and set

$$\sigma_*(t) = \int_E v(\zeta)\sigma_0(t)v(\zeta)^{-1} d\zeta \,. \tag{II.2}$$

Then

$$v(\zeta_1)\sigma_*(t)v(\zeta_1)^{-1} = \int_E v(\zeta_1)v(\zeta)\sigma_0(t)v(\zeta)^{-1}v(\zeta_1)^{-1} d\zeta$$

and invariance of the measure means that this is the original integral defining $\sigma_*(t)$. Thus σ_* commutes with all $v(\zeta)$. Since $v(\zeta)^{-1} = v(\zeta)^*$, we see that $\sigma_0(t) \geq 0$ implies $\sigma_*(t) \geq 0$.

To show that σ_* is a storage function for all of E we need only show, by part (a), that σ_* is a storage function for a single system $[A_{\zeta_0}, B_{\zeta_0}, C_{\zeta_0}]$. Since σ_0 satisfies (I.2) for all systems we see (as in part (a)) that for each $\zeta \in E$

$$\sigma_\zeta = v(\zeta)\sigma_0 v(\zeta)^{-1}$$

satisfies (I.2) for the fixed system $[A_{\zeta_0}, B_{\zeta_0}, C_{\zeta_0}]$. Now the left side of (I.2) is linear in σ except for the quadratic term $-\sigma B\Gamma\Gamma^* B^*\sigma$ which is concave. Thus the left side of (I.2) is concave in σ. Consequently if σ_1 and σ_2 satisfy (I.2) then so does any convex combination. Since $\sigma_* = \int_E \sigma_\zeta d\zeta$ is the limit of convex combinations of solutions to (I.2), it follows that σ_* satisfies (I.2) for $[A_{\zeta_0}, B_{\zeta_0}, C_{\zeta_0}]$ as was to be shown.

PART III. STABILITY OF A FAMILY OF SYSTEMS

In this part we change viewpoints and discuss the problem of finding weights Γ and β so that all systems in SYS^K are dissipative. This type of problem would arise when one is not using dissipativity to insure performance, but rather to insure stability. Since freedom in β has an influence comparable to freedom in Γ the problem seems underdetermined. Thus, to add structure, we analyze the following problem:

Given K and β find a weight Γ so that all systems in SYS^K are dissipative.

For example, if the systems SYS^K and weights Γ and β were time invariant, then the problem (for single input single output systems) is equivalent to showing that the frequency response functions of systems in SYS^K are uniformly bounded by some number (Γ/β in this setting).

We are particularly interested in the case that *there is a single σ for all systems.* Henceforth we shall assume that the extreme points E of K satisfy EEP and form a commutative compact group. This is the situation which occurs with the Bieberbach problem. We shall find that the extra freedom in Γ permits considerable simplification of the results in part II.

De Branges analyzed the single σ case of some problems, and this section is closest to section I130 of [**VN**]. Our procedure is to consider a particular system $[A_{\zeta_0}, B_{\zeta_0}, C_{\zeta_0}] \in E$ and to construct a storage function $\sigma_* \geq 0$ for it which commutes with all $v(\zeta)$. This will involve some further assumptions on the systems in E. The results of part II show that if we can find a Γ and such a σ_*, then σ_* is a storage function for all systems in SYS^K (with respect to Γ). Conditions for the existence of Γ will be given in Theorem III.1.

The construction of σ is based on writing the Riccati equation (I.2) in the standard way as a Hamiltonian system. Recall this formulation is given by the equations (I.10x) and (I.10p)

$$\frac{dx}{dt} = Ax + B\Gamma\Gamma^* B^* p$$

$$\frac{dp}{dt} + A^* p + C^* \beta^{*-1} \beta^{-1} C x = 0$$

and a solution x, p provides a solution σ of (I.2) by the relation $p = \sigma x$.

The idea is to pick x_*, u_* functions of time so that x_* is obtained from u_* acting as a control on the system

(III.1)
$$\frac{dx_*}{dt} = A_{\zeta_0} x_* + B_{\zeta_0} u_* \, .$$

Call x_*, u_* a *control pair.* Now with $x = x_*$ in (I.10p), a solution p_* to

$$\frac{dp}{dt} + A^*_{\zeta_0} p + C^*_{\zeta_0} \beta^{*-1} \beta^{-1} C_{\zeta_0} x_* = 0$$

can be obtained once an initial (or final) condition $p_{00} \in X$ is specified. At this point we have a triple x_*, u_*, p_*.

Now we construct σ_*. We do not use the averaging formula (II.2) of Lemma II.2(b) directly, but instead do something more computable. Since E is a compact commutative group we can pick an eigenbasis e_k for $v(\zeta)$, thus

$$v(\zeta)e_k = \alpha_k(\zeta)e_k \quad \text{for some scalars } \alpha_k(\zeta).$$

Naturally σ_* commutes each $v(\zeta)$ is assured if σ_* is diagonal in the e_k basis for the state space X. It is equivalent if multiplicity of α_k is one. It is easy to build a diagonal σ_* † from a given x_* and p_* by the formula:

$$\sigma_*(t)_k = p_*(t)_k / x_*(t)_k$$

for its diagonal entries. To check that σ_* is positive definite one simply checks if each σ_{*k} is positive.

Now that we have indicated how one would find $\sigma_* \geq 0$ commuting with $v(\zeta)$ the next step is to check if there is some Γ so that it satisfies (I.2). To describe some results define Λ by

$$\text{(III.2)} \qquad \Lambda(\sigma(t),t) = -\frac{d\sigma}{dt}(t) - \sigma(t)A_{\zeta_0} - A^*_{\zeta_0}\sigma(t) - C^*_{\zeta_0}\beta(t)^{*-1}\beta(t)^{-1}C_{\zeta_0} .$$

† This is the same type of construction one finds in the spectral theorem for selfadjoint operators. It usually looks like this: $\phi_\mu := \int_E v(\zeta)d\mu(\zeta)$. For each time t, define $\sigma_*(t)$ by

$$\sigma_*(t)\left(\phi_\mu x_*(t)\right) = \phi_\mu\left(p_*(t, p_{00})\right)$$

for all μ measures of total mass 1. It is well defined on $\{\phi_\mu x_* : \mu \text{ is a measure of mass } 1\}$ which one assumes is dense in X.

Thus, in terms of Λ, the Riccati inequality (I.2) says

$$\Lambda(\sigma_*, t) - \sigma_* B_{\zeta_0} \Gamma\Gamma^* B_{\zeta_0}^* \sigma_* \geq 0 \ . \tag{III.3}$$

Clearly $\Lambda(\sigma_*, t) \geq 0$ is a necessary condition for a Γ to exist such that σ_* is a storage function. The next theorem describes under what additional assumptions $\Lambda \geq 0$ is also a sufficient condition.

THEOREM III.1. *Assume EEP where E is a compact commutative group. For all t suppose that $\sigma_*(t) \geq 0$ is an invertible operator that commutes with $v(\zeta)$ (where $v(\zeta)$ is defined with respect to a fixed ζ_0). Let β be fixed and assume that $\Lambda \geq 0$. Then*

(i) If for each t Range $\Lambda(\sigma_*, t) \supset$ Range $\sigma_*(t) B_{\zeta_0}$, then an invertible Γ exists so that σ_* is a storage function for all systems in SYS^K.

Moreover,

(ii) If additionally for a particular x_* we have that $\Lambda x_* \in$ Range $\sigma_* B_{\zeta_0}$, then we may take Γ, satisfying the conclusions of (i), so that u_* defined by $u_* = \Gamma\Gamma^* B_{\zeta_0}^* \sigma_* x_*$ makes x_*, u_* a control pair. In other words, u_* is an optimal feedback law (the least dissipative input). Indeed the system $[A_{\zeta_0}, B_{\zeta_0}, C_{\zeta_0}]$ conserves energy (no energy is dissipated) on the trajectory x_*, u_*:

$$\int_s^r \|u_*(t)\|_{\Gamma(t)}^2 dt = \|x_*(r)\|_r^2 - \|x_*(s)\|_s^2 + \int_s^r \|\beta(t)^{-1} C_{\zeta_0} x_*(t)\|^2 \, dt \ . \tag{III.4}$$

COROLLARY III.2. *Suppose $[A, B, C]$ is a time varying system in SYS^K. Suppose*

E satisfies EEP and that ζ_0, x_*, u_* satisfy the hypothesis of Theorem III.1. Then

(III.5)
$$\|C_s^r u_*\|_r^2 \leq \|x_*(r)\|_r^2 - \|x_*(s)\|_s^2 + \int_s^r \|\beta(t)^{-1} C_{\zeta_0} x_*(t)\|^2 \, dt$$

where C_s^r is the controllability operator for the system $[A, B, C]$.

Proof. Combine Theorem III.1(ii) with estimate (I.3).

We use the following standard facts about range inclusion to prove Theorem III.1:

Given $M \geq 0$ and P bounded operators there is a bounded $Q \geq 0$ satisfying

(III.6) $\quad M \leq PQP^*$ *iff* *Range* $M \subset$ *Range* P

(III.7) $\quad M \geq PQP^*$ *and Q is invertible* *iff* *Range* $M \supset$ *Range* P

(III.8) $\quad M = PQP^*$ *and Q is invertible* *iff* *Range* $M =$ *Range* P

(III.9)

$Mx = PQP^*x$ *and Q is invertible* *iff* *Range* $M \supset$ *Range* P *and* $Mx \in$ *Range* P ∎

These facts follow because range inclusion makes right and left inverses of $M^{1/2}$ and P defined in appropriate places. For example, (III.6) is true because a Moore-Smith-Penrose left inverse [**H**] P^l of P is defined on Range $M \subset$ Range P, and so $P^l M P^{l^*}$ can be taken as Q.

Proof Theorem III.1. (i) Assume that Range $\Lambda \supset$ Range $\sigma_* B_{\zeta_0}$. Since $\Lambda \geq 0$ we may use (III.7) to obtain an invertible $Q \geq 0$ such that $\Lambda \geq \sigma_* B_{\zeta_0} Q B^*_{\zeta_0} \sigma_*$. Now factor Q as $\Gamma\Gamma^*$ to see that (III.3) and hence the Riccati inequality (I.2) holds for σ_* on the system $[A_{\zeta_0}, B_{\zeta_0}, C_{\zeta_0}]$ with respect to Γ. Since σ_* was assumed to commute with all $v(\zeta)$, Lemma II.2(a) says that σ_* is a storage function for SYS^K in the $\| \|_\Gamma$ 'metric'. Finally, note that Γ is invertible because Q is invertible.

(ii) Now assume additionally that $\Lambda x_* \in$ Range $\sigma_* B_{\zeta_0}$. Proceed exactly as in (i), except using (III.9) instead of (III.7), to obtain a Γ satisfying the conclusions of (i) and such that

$$\Lambda x_* = \sigma_* B_{\zeta_0} \Gamma\Gamma^* B^*_{\zeta_0} \sigma_* x_* \ .$$

To show $u_* = \Gamma\Gamma^* B^*_{\zeta_0} \sigma_* x_*$ makes x_*, u_* a control pair we must verify (III.1). We have that

$$\begin{aligned} \sigma_*(A_{\zeta_0} x_* + B_{\zeta_0} u_*) &= \sigma_* A_{\zeta_0} x_* + \sigma_* B_{\zeta_0} \Gamma\Gamma^* B^*_{\zeta_0} \sigma_* x_* \\ &= \sigma_* A_{\zeta_0} x_* + \Lambda x_* \ . \end{aligned}$$

So using the definition (III.2) of Λ and equation (I.10p) we get that

$$\begin{aligned} \sigma_*(A_{\zeta_0} x_* + B_{\zeta_0} u_*) &= \sigma_* A_{\zeta_0} x_* - \frac{d\sigma_*}{dt} x_* - \sigma_* A_{\zeta_0} x_* - A^*_{\zeta_0} \sigma_* x_* \\ &\quad - C^*_{\zeta_0} \beta(t)^{*-1} \beta(t)^{-1} C_{\zeta_0} x_* \\ &= -\frac{d\sigma_*}{dt} x_* - A^*_{\zeta_0} \sigma_* x_* - C^*_{\zeta_0} \beta(t)^{*-1} \beta(t)^{-1} C_{\zeta_0} x_* \\ &= -\frac{d\sigma_*}{dt} x_* + \frac{d\sigma_* x_*}{dt} \\ &= \sigma_* \frac{dx}{dt}. \end{aligned}$$

Since σ_* is invertible we conclude that $\frac{dx}{dt} = A_{\zeta_0} x_* + B_{\zeta_0} u_*$ as required.

It remains to verify the energy conservation formula (III.4). Now (I.2) holds with equality which is equivalent to (I.5) holding with equality, and in this notation it says

$$\|u_*(t)\|^2_{\Gamma(t)} - \|\beta(t)^{-1}C_{\zeta_0}x_*(t)\|^2 - \frac{d}{dt}\,\|x_*(t)\|^2_t = 0\ .$$

Integrating this completes the proof.

PART IV. TOWARD THE MILIN INEQUALITIES: A DEGENERATE CASE

This section indicates how the Bieberbach problem fits together with the theory in sections I to III. It is an informative illustration of the methods for obtaining uniform bounds on the behavior of a family of systems.

a. Bieberbach Background

The Bieberbach conjecture concerns the class S of functions univalent (analytic and one-to-one) in the unit disk Δ with expansion

$$z + \sum_{n=2}^{\infty} a_n z^n\ . \tag{IV.1}$$

In 1916 Bieberbach conjectured that if $f(z) \in S$ has expansion (IV.1) then

$$|a_n| \leq n \tag{IV.2}$$

for each $n \geq 2$. Further if equality holds for a single n then $f(z)$ is, up to a rotation, the Koebe function $k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} nz^n$. Many prominent mathematicians attempted to solve this problem in the years that followed. The inequality (IV.2) was shown to hold for small values of n ($n \leq 6$) and non-sharp uniform bounds for $|a_n|/n$ were proved. It was not until 1984 that the problem was solved completely by de Branges [**dB**] using a highly original approach. See [**dB1-4**] for the main ideas in the subject. For other interesting arguments see [**FP**].

De Branges did not show Bieberbach's conjecture directly, he in fact proved a stronger set of inequalities known as the Milin, or Lebedev-Milin, inequalities. To describe these let $f(z) \in S$ and define the numbers c_k by

$$\log \frac{f(z)}{z} = \sum_{k=1}^{\infty} c_k z^k \qquad \text{for } z \in \Delta \ .$$

The Milin inequalities are

$$(IV.3) \qquad \sum_{k=1}^{n} k(n+1-k)|c_k|^2 \leq 4 \sum_{k=1}^{n} \frac{n+1-k}{k} \quad \text{for } n = 1, 2, \ldots$$

and their validity for each $f \in S$ implies the Bieberbach conjecture. It is not difficult to show via a limiting argument [**VN2**, E10] that the Milin inequalities need only be verified for each bounded $f(z) \in S$ to show that they hold for all $f(z) \in S$.

Systems theory enters the study of univalent functions through the Löwner differential equation. There are various versions of this equation, the one appropriate here is

$$\text{(IV.4)} \qquad \frac{dx}{dt}(t, z) = \Omega(t) x(t, z)$$

where

$$\Omega(t) = \varphi(t, \cdot) z \frac{d}{dz}$$

with some $\varphi(t, \cdot)$ which for each t is analytic on the disk with positive real part and has value 1 at $z = 0$ [**P**]. For fixed t the solution $x(t, z)$ of (IV.4) is a univalent function in z. Moreover for an arbitrary $f(z) \in S$ there exists $\varphi(t, \cdot)$ such that $f(z)$ is the initial value of the solution $x(t,z)$ of (IV.4). Thus theorems about all Löwner equations give theorems about all functions in S.

We denote propagation operators for this system of evolution equations by B_{rs} which can be regarded as univalent functions of the disk mapping into the disk satisfying

$$\frac{\partial B_{rs}}{\partial r} = \varphi(r,\cdot) z \frac{\partial B_{rs}}{\partial z} \qquad \frac{\partial B_{rs}}{\partial s} = -B_{rs}\varphi(s, B_{rs}) \tag{IV.5}$$

Now we shift from the classical to a more function analytic view. A natural Hilbert space for these problems is the space of analytic functions $f(z)$ defined on the unit disk and normalized by $f(0) = 0$ with the Dirichlet inner product:

$$(f,g) = \sum_{n\geq 1} n f_n \overline{g_n}$$

where $f(z) = \sum f_n z^n$ and $g(z) = \sum g_n z^n$. Let $\mathcal{G}$ be the Hilbert space defined by

$$\mathcal{G} = \{f : \sum_{n\geq 1} n|f_n|^2 < \infty\} .$$

We shall work with composition operators on $\mathcal{G}$

$$T_{rs} f = f \circ B_{rs} \quad \text{for } f \in \mathcal{G}, \tag{IV.6}$$

induced by the univalent maps B_{rs}. At this point we start dealing formally since $T_{rs}f$ is an analytic function on a Dirchlet space sitting over the image of the univalent map B_{rs}. The composition operators T_{rs} satisfy the differential equation

$$\frac{\partial T_{rs}}{\partial r} = \Omega(r) T_{rs} \qquad \frac{\partial T_{rs}}{\partial s} = -T_{rs}\Omega(s) \tag{IV.7}$$

which is usual except [**VN**] take the system to run backward in time, so $r < s$. Running backwards in time makes the B_{rs} conformal maps of the unit disk *into* the unit disk and de Branges many years ago observed that the T_{rs} are contractions on $\mathcal{G}$.

Using the subordination principle [**L**] and the Löwner equation it can be shown [**VN2**, D10, D20, and E10] that if $f(z) \in S$ is bounded then there is a $\Omega(t)$ corresponding to f and a number b (depending on $\sup_{z\in\Delta} |f(z)|$) such that when the Löwner equation is initialized at the identity map z its solution at time b has value f. This is a type of reachability. If $f \in S$ is not bounded it is approximatable by bounded f. Thus a complete understanding of Löwner equations would produce a complete understanding of univalent functions.

b. The formal systems setup

The idea is to place the Löwner differential equation (IV.4) into the framework of a system of equations and use the results developed in parts I–III to obtain estimates valid for all such systems. When we specialize these estimates to the particular Löwner system corrsponding to a given bounded $f(z) \in S$ we shall obtain (for an appropriately choosen weight function σ) the Milin inequalities for that particular $f(z)$.

Not all that we have done is required to prove the Milin inequalities. Although one needs greater generality in the direction of infinite dimensions and time varying domains. Indeed the Milin inequalities pertain to the degenerate case where the system has no output. We are interested only in the special case of (I.9) which accounts for energy transferred between inputs and states.

Essentially all formulas and proofs we need can be obtained by setting C to 0. Here one ignores the IO dissipative property and uses only the Bounded Real Lemma and (I.2) with $C = 0$.

Alternatively, we could fix C thereby having a legitimate output from the system, but weight the output from the system with $\beta(t)^{-1}$ near 0. That is, the energy of the output is very small even if the output is big. For the Bieberbach problem all we need is Corollary III.2 and in the limit as $\beta(t)^{-1} \to 0$ the key inequality (III.5) in Corollary III.2 becomes

$$\|\mathcal{C}_s^r u_*\|_r^2 \leq \|x_*(r)\|_r^2 - \|x_*(s)\|_s^2 \tag{IV.8}$$

where x_*, u_* are an optimal control pair for $[A_{\zeta_0}, B_{\zeta_0}]$ as described in Corollary III.2. This is the end of general systems theory in this paper. What remains is special to Löwner systems.

The operator Ω is accretive on $\mathcal{G}$ if and only if $\varphi(t,\cdot)$ has positive real part [**VN2**, D110]. Since any positive real part function has a Herglotz representation, there exists for each t a probability measure μ_t on the unit circle such that

$$\varphi(t,z) = \int_{|\zeta|=1} \frac{1+\zeta z}{1-\zeta z}\, d\mu_t(\zeta)\,. \tag{IV.9}$$

Thus it is plausible (and one can check) that the extreme points of the operators $\Omega(t) = \varphi(t,\cdot) z \frac{d}{dz}$ are

$$\Omega_\zeta = \frac{1+\zeta z}{1-\zeta z} \zeta z \frac{d}{dz} \quad \text{for } |\zeta| = 1\,.$$

Observe that the operator Ω_ζ can be generated from the fixed operator Ω_1 by a unitary rotation operator $v(\zeta)$ defined by

$$v(\zeta)f(z) = f(\zeta z)\ ,$$

that is

$$\Omega_\zeta = v(\zeta)\Omega_1 v(\zeta)^{-1}.$$

Thus we take our class of extreme systems E to be

$$\frac{dx}{dt}(t) = \Omega_\zeta - v(\zeta)u(t) \quad \text{for } |\zeta| = 1,$$

where $u(t) \in \mathcal{G}$ for each t. Having done this we have insured that E satisfies EEP. Set SYS^K to be the time varying convex hull of E as before.

This is a bit different from [**VN**] (see I180) who take E to be

$$\frac{dx(r)}{dr} = \Omega_\zeta x(r) - g(r)$$

where $g(r) \in \mathcal{G}$. Then they take convex combinations of systems E and corresponding convex combinations of inputs g to obtain their version (C110) of our Corollary III.2 and (IV.8). For the Bieberbach example, (C110) and (IV.8) give the same answer, while for systems with noninvertible B they do not. Corollary III.2 seems conceptually simpler and has a standard engineering description given in parts II and III.

One sees from (IV.6) that the statespace at time t is all analytic functions on the domain $B_{t0}(\Delta)$. Therefore the statespace is time varying. Also the Löwner equations are accretive rather than dissipative which is why [**VN**] run them backward in time. This paper is aimed at finite dimensional systems and so our brief sketch of the Bieberbach structure proceeds formally by suppressing domain issues and converting forward time to backward time without filling in many details. These issues are all carefully covered in [**VN**].

c. bounding SYS^K

In order to obtain the estimate (IV.8) of Corollary III.2 mentioned earlier we must construct a $\sigma_* \geq 0$ and a Γ such that the hypotheses of Theorem III.1 hold. We have already shown EEP and taken $\zeta_0 = 1$. We proceed in the construction of σ_* as outlined in part III. For the Bieberbach problem we take

$$x_*(t,z) = -2\log(1+z)$$

a constant in time. (The choice of x_* is motivated by the fact that $x_*(t,z) = \log \frac{f(z)}{z}$ where $f(z) = \frac{z}{(1+z)^2}$ is the Koebe function.)

A control u_* satisfying (III.1) is easily obtained since, in the notation of part III, $B_{\zeta_0} = -v(\zeta_0) = -I$ is invertible. Specifically we take

$$u_* = \Omega_1 x_* = \frac{1+z}{1-z} z \frac{d}{dz} x_* = -2\frac{z}{1-z}.$$

Since $C = 0$, the Hamiltonian for p is

$$\text{(IV.10)} \qquad \frac{dp}{dt} + \Omega_1^* p = 0\ .$$

Recall that a solution p_* of this equation shows the existence of a σ_* satisfying (I.2) where p_*, x_*, and σ_* are related by $p_* = \sigma_* x_*$.

In order to insure that σ_* commutes with $v(\zeta)$ we wish to construct σ_* to be diagonal in the eigenbasis of $v(\zeta)$. The eigenbasis for $v(\zeta)$ is given by $e_k = z^k$, hence the expansion of an analytic function in this basis is the Fourier series expansion. Thus we shall attempt to construct σ_* to be a diagonal matrix in the e_k basis whose diagonal entries σ_{*_k} are defined by the formula

$$\sigma_{*_k} = \frac{p_{*_k}}{x_{*_k}}$$

where p_{*_k} and x_{*_k} are the $k-$th Fourier coefficients of p_* and x_* respectively.

It turns out that equation (IV.10) is most conveniently expressed by taking its inner product not with e_k, but rather with $\frac{e_k}{k} - \frac{e_{k+1}}{k+1}$. Doing this we see that (IV.10) is equivalent to the system of equations

$$\frac{dp_n}{dt} - \frac{dp_{n+1}}{dt} + np_n + (n+1)p_{n+1} = 0 \qquad n \geq 1.$$

Since x_{*_n} is easily calculated to be $2\frac{(-1)^n}{n}$, the existence of a diagonal σ_* satisfying (I.2) is equivalent to solving the system of equations

$$\text{(IV.11)} \qquad \frac{\frac{d\sigma_n}{dt}}{n} + \frac{\frac{d\sigma_{n+1}}{dt}}{n+1} + \sigma_n - \sigma_{n+1} = 0 \qquad n \geq 1.$$

These are the de Branges equations [**dB**, p.141].

Now we must determine if there are solutions σ of (IV.11) such that Λ given by (III.2) (with $C = 0$) is non-negative and $\sigma \geq 0$. The Λ in [**VN**] is actually $-\Lambda$ in (III.2) since at this point we are running the system backward in time. The conditions $-\Lambda \geq 0$ and $\sigma \geq 0$ can be formulated, as is done by [**VN**] (see I200), in terms of Jacobi matrices. We refer the reader to [**VN2**] for the details and just provide that results here: $-\Lambda \geq 0$ is equivalent to $\frac{d\sigma_k}{dt} < 0$ for each k [**VN2**, E60], and $\sigma(t) \geq 0$ occurs if the initial values $\sigma_k(0)$ form a convex sequence (i.e. $\sigma_n(0) - \sigma_{n+1}(0) \geq \sigma_{n+1}(0) - \sigma_{n+2}(0)$). Further, it can be shown with some effort [**VN2**], that there exists a solution with $\frac{d\sigma_k}{dt} < 0$ and with the initial (convex) values

$$\sigma_k(0) = \begin{cases} n-k+1, & k = 1, \ldots, n \\ 0 & k > n \end{cases} \tag{IV.12}$$

for any $n \geq 1$. These initial weights are precisely what is needed in (IV.8) to obtain the Milin inequalities.

One must check that the conditions of Theorem III.1 hold for Λ so that a Γ exists for which (III.5) holds. [**VN**] are murky on this, but assert that (III.5) holds.

d. Estimates on states of SYS^K gives the Milin inequalities

The Milin inequalities are gotten simply by plugging what we already have into the (backward time version of the) controllability matrix inequality (IV.8) with $\zeta_0 = 1$. The formal backward time controllability operator for a system

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ &= \left(\int \Omega_\zeta(t) d\mu_t(\zeta)\right) x \; + \left(\int -v(\zeta) d\mu_t(\zeta)\right) u \end{aligned}$$

in SYS^K is (in backward time)

$$\begin{aligned} [\mathcal{C}_r^t u](t) &= \int_r^t T_{rs}(-B(s))u(s)\ ds \\ &= \int_r^t T_{rs} \int_E v(\zeta)u(s) d\mu_s(\zeta)\ ds\ . \end{aligned}$$

Recall that $u(s)$ is an analytic function in $\mathcal{G}$ and that $v(\zeta)$ is a rotation operator. Thus the composition property (IV.6) of T_{rs} gives

$$[\mathcal{C}_r^t u](t) = \int_r^t \int_E u(s) \circ \zeta B_{rs}\ d\mu_s(\zeta)\ ds\ .$$

Here the system is 'starting' at $x(r) = 0$. Now we start comparing the system $[A, B]$ to the reference system $[A_{\zeta_0}, B_{\zeta_0}]$ by applying the controllability operator to the 'optimal' input $u_* = -2\frac{z}{1-z}$ for the reference system. We get

$$\begin{aligned}
[\mathcal{C}_r^t u_*](t) &= -2\int_r^t \int_E \frac{\zeta B_{rs}}{1-\zeta B_{rs}} d\mu_s(\zeta)\ ds \\
&= -2\int_r^t \int_E \frac{1}{2}\left[\frac{1+\zeta B_{rs}}{1-\zeta B_{rs}} - \frac{1-\zeta B_{rs}}{1-\zeta B_{rs}}\right] d\mu_s(\zeta)\ ds \\
&= \int_r^t \left[1 - \int_E \frac{1+\zeta B_{rs}}{1-\zeta B_{rs}} d\mu_s(\zeta)\right]\ ds\ .
\end{aligned}$$

Using the defintion (IV.9) of $d\mu_s$ and then applying the Löwner equations (IV.5) we calculate

$$\begin{aligned}
[\mathcal{C}_r^t u_*](t) &= \int_r^t [1 - \varphi(s, B_{rs})]\ ds \\
&= \int_r^t \left[1 + \frac{1}{B_{rs}}\frac{\partial B_{rs}}{\partial s}\right]\ ds \\
&= t - r + \log B_{rt} - \log B_{rr} \\
&= \log \frac{B_{rt}}{e^{r-t} B_{rr}}\ .
\end{aligned}$$

For a given bounded $f(z) \in S$, the B_{rs} can be taken (by choice of $A_\zeta = \Omega_\zeta$) [**VN2**, E10] to satisfy initial, final, and boundary conditions

$$B_{ss} \equiv z\ , \quad B_{0b} = e^{-b} f\ , \quad B_{rs}(0) = 0\ , \quad B'_{rs}(0) = e^{r-s}$$

for $0 \le r \le s \le b$. Thus

$$[\mathcal{C}_r^t u_*](t) = \log \frac{B_{rt}}{B'_{rt}(0) z}$$

and, in particular

$$[\mathcal{C}_0^b u_*](b) = \log \frac{f(z)}{z}\ .$$

Now we use this in the backward time version

$$\|\mathcal{C}_r^t u_*\|_r^2 \le \|x_*(r)\|_r^2 - \|x_*(t)\|_t^2 \tag{IV.13}$$

of (IV.8). Let $|c_k|$ denote the Fourier coefficients of $\log \frac{f(z)}{z}$ as before, and apply (IV.13) with $r = 0$ and $t = b$ to get

$$\sum_{k\ge 1} k\sigma_k(0)|c_k|^2 \le \sum_{k\ge 1} \sigma_k(0)k\left(\frac{2}{k}\right)^2 - \sum_{k\ge 1}\sigma_k(b)k\left(\frac{2}{k}\right)^2 \le 4\sum_{k\ge 1}\frac{\sigma_k(0)}{k}$$

valid for any $\sigma \ge 0$ as constructed in section c. In particular we may take σ initialized by (IV.12) to obtain

$$\sum_{k=1}^{n} k(n+1-k)|c_k|^2 \le 4\sum_{k=1}^{n}\frac{n+1-k}{k}$$

for any $n \ge 1$, which is exactly the Milin inequalities (IV.3) for $f(z)$. This proves the Milin inequalities when $f(z) \in S$ is bounded, and hence (by a limiting argument) for all $f(z) \in S$.

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References

[AV] Brian D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis: a Modern Systems Theory Approach*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1973.

[B] S. Boyd, *Control System Analysis and Synthesis via Linear Matrix Inequalities*, ACC (plenary lecture, San Fransisco), 1993, pp. 2147–2154.

[dB] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), pp. 137–152.

[dB1] L. de Branges, *Powers of Riemann mapping functions* in *The Bieberbach conjecture*, Proc of the Symposium on the occasion of the proof of the Bieberbach conjecture, AMS, 1986, Providence R.I.

[dB2] L. de Branges, *Unitary linear systems whose transfer functions are Riemann mapping functions* Operator Thy Advances and Appl. vol 19 Birkhauser, Bassel 1986 ,pp105-124

[dB3] L. de Branges, *Underlying concepts in the proof of the Bieberbach conjecture* Proc of the International Congress of Math, AMS Providence R.I. 1987, pp 25-42.

[dB4] L. de Branges, *Square summable power series* to appear

[BO] T. Basar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, Academc Press, Math in Sci. & Eng. series **160** (1982).

[FP] C. FitzGerald and Chr. Pommerenke, *The de Branges theorem on univalent functions*, Trans. Amer. Math. Soc. **290** (1985), pp. 683–690.

[H] Magnus R. Hestenes, *Inversion of matrices by biorothogonalization*, J. Soc. Indust. Appl. Math. **6** (1958), p. 84.

[HM] D. J. Hill and P. J. Moylan, *Dissipative Dynamical Systems: Basic Input-Output and State Properties*, J. Franklin Inst. **309** (1980) no. 5, pp. 327–357.

[L] J. E. Littlewood, *On inequalities in the theory of functions*, Proc. Lond. Math. Soc. **2** (1925), pp.481–519.

[LD] Lu and J. Doyle, H^∞ *Control of nonlinear systems: a convex characterization* (preprint).

[P] Chr. Pommerenke, *Univalent functions*, Vandehoeck and Ruprecht, Göttingen, 1975, pp. 156–164.

[vS] A. J. van der Schaft, L_2 *Gain Analysis of Nonlinear Systems and Nonlinear* H^∞ *Control*, IEEE Trans. Auto. Control (to appear).

[VN] V. I. Vasyunin and N. K. Nikol'skii, *Quasiorthogonal decompositions with respect to complementary metrics, and estimates of univalent functions*, Leningrad Math. J. **2** (1991) no. 4, pp. 691–764.

[VN2] V. I. Vasyunin and N. K. Nikol'skii, *Operator-valued measures and coefficients of univalent functions* (preprint).

[W] J. C. Willems, *Dissipative dynamical systems*, Arch. Rat. Mech. Anal. **45** (1972), Part I pp. 321–351, Part II pp. 352–393.