**RULES FOR COMPUTER SIMPLIFICATION OF THE FORMULAS IN OPERATOR MODEL THEORY AND LINEAR SYSTEMS**

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Dedicated to Moshe Livsic on the occasion of his 70th birthday

This article formulates and treats questions in operator theory arising from computer simplification of formulas commonly found in the study of operator models. Operator model theory originated with Moshe Livsic and subsequently became one of the main branches of operator theory. In studying a particular operator model polynomials in certain expressions occur repeatedly. This makes it a natural area for exploring computer algebra simplification.

The purpose of a simplification theory is to provide a means for replacing complex expressions by expressions which are "simpler" in some sense. The main task is to obtain a list of rules each of which replaces a "complicated" monomial which occurs in an expression by a sum of "simpler" monomials. To simplify such an expression one applies the rules to the expression until no further reduction is possible. The result is called an N-Form (normal form) for the original expression. The reduction of an expression to an N-Form can be easily implemented on a computer. This article provides a collection of reduction rules for expressions which arise in the Nagy-Foias operator model. Our simplifying rules were obtained by applying an algorithm for computing a Grobner basis for an ideal in a polynomial ring. It is applied to the ideal generated by a set of fundamental relations which obviously hold for NF calculations. We conjecture that all appropriate relations are in this ideal. The algorithm produces an infinite set of rules in the NF case. The traditional operator theorist's functional calculus is used to produce a nice formulation of the rules as a finite set.

If a set of generators is a Grobner Basis, then the reduction to an N-form has very nice properties; the N-form is independent of the order in which rules are applied and equality of N-forms can be used as a test of equivalence of expressions mod the ideal. We have established that our rules form a Grobner Basis for several situations. A proof outline and computer tests provide strong evidence that this is true for the NF case as well.

The results here have potential applications to engineering systems theory since this algebraic structure occurs in formulas arising in $H^\infty$ control. Also since the entries of a unitary 2x2 block matrix have algebraic structure similar to the NF model, this applies to 'all pass' functions from engineering.

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## I. INTRODUCTION

Our goal is to find Gröbner bases for polynomials in four different sets of expressions:

| | | | | | | |
|---|---|---|---|---|---|---|
| (RESOL) | $x\,,$ | $x^{-1}\,,$ | $(1-x)^{-1}$ | | | |
| (EB) | $x\,,$ | $x^{-1}\,,$ | $(1-xy)^{-1}$ | | | |
| | $y\,,$ | $y^{-1}\,,$ | $(1-yx)^{-1}$ | | | |
| (preNF) | (EB) | plus | $(1-x)^{-1}$ | and | $(1-y)^{-1}$ | |
| (NF) | (preNF) | plus | $(1-xy)^{1/2}$ | and | $(1-yx)^{1/2}$ | |

Most formulas in the theory of the Nagy-Foias operator model [NF] are polynomials in these expressions where $x = T$ and $y = T^*$. Complicated polynomials can often be simplified by applying "replacement rules". For example, the polynomial

$$(1-xy)^{-2} - 2xy(1-xy)^{-2} + xy^2\,(1-xy)^{-2} - 1$$

simplifies to $0$. This can be seen by three applications of the replacement rule $(1-xy)^{-1}xy \longrightarrow (1-xy)^{-1} - 1$ which is true because of the definition of $(1-xy)^{-1}$.

A replacement rule consists of a left hand side (LHS) and a right hand side (RHS). The LHS will always be a monomial. The RHS will be a polynomial whose terms are "simpler" (in a sense to be made precise) than the LHS. An expression is reduced by repeatedly replacing any occurrence of a LHS by the corresponding RHS. The monomials will be well-ordered, so the reduction procedure will terminate after finitely many steps. Our aim is to provide a list of substitution rules for the classes of expressions above. These rules, when implemented on a computer, provide an efficient automatic simplification process.

We discuss and define the ordering on monomials later. Such an ordering typically is chosen to reflect the intuitive idea that complicated expressions are to be replaced by simpler ones.

We find that for cases (RESOL) and (EB) the ideal of relations (considered as a polynomial ring in the non-commuting atomic expressions above) is finitely generated. A finite Gröbner basis is obtained using a modification of the Buchberger algorithm due to F. Mora [Mora]. The ideal of relations in the other two cases can be shown not to be finitely generated—and the Mora algorithm does not terminate. We find, however, that the set of relations generated by the algorithm ultimately consists of a collection of classes of general relations which, with some particular relations, forms an infinite Gröbner Basis.

We introduce a formulation for our results in terms of an algebra which contains the operator expressions of interest. This formulation is very intuitive to a functional analyst in that it incorporates the operator functional calculus. An advantage here is that the infinite set of replacement rules obtained by applying the Basis Algorithm can be expressed by a finite set in the new algebra.

It will be interesting to see if other areas of operator theory behave the same way since it is natural to try what we have done on many of them. We expect that a major challenge for operator theorists will be to to take the infinite lists of rules which are appropriate for a particular field and find parameterizations which make them a natural looking finite list.

Much of the technique used here is algebra of the kind common in operator theory, while the non-commutative Gröbner Basis theory is not familiar in the operator community. Since this article is intended to be accessible to someone who is not familiar with this theory we shall outline basics. This article, however, is not intended to be a complete survey of the field.

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## II. THE REDUCTION AND BASIS ALGORITHMS

### A. Commuting Variables

#### 1. The Reduction Process.

We impose an ordering (a "term order") on the monomials in the ring of polynomials in several commuting variables $(x_1, \ldots, x_n)$ over a field. Once such an ordering is imposed, we obtain a reduction process:

Let $F = \{f_1, \ldots, f_k\}$ be a set of polynomials in commuting variables. Using the given ordering, each can be written with the monomials in descending order. Now let $f$ be any polynomial. We say that $f$ is reducible to $g$ with respect to $F$ if $g = f - cx^\alpha f_i$ where $c$ and $\alpha$ are chosen so that the leading term of $cx^\alpha f_i$ coincides with one of the terms of $f$. The effect is to replace a term of $f$ with terms of lower order. A given $f$ can be "simplified" by iterating this process until a polynomial is obtained which can no longer be reduced (none of its terms are divisible by any of the leading terms of the $f_i$). We call this a normal form (N-form) for $f$ with respect to the process. Notice that any polynomial obtained from $f$ by reduction is congruent to $f$ mod the ideal generated by $F$.

### 2. The Gröbner Basis Algorithm.

Buchberger [Buch] and Hironaka [Hiron] independently showed that every ideal in a polynomial ring has a basis for which the reduction process is well-behaved. Buchberger provided a constructive procedure for obtaining such a basis from an arbitrary system of generators for the ideal.

> **Definition**. A set, $F$, of polynomials is called a Gröbner Basis for the ideal, I, it generates if it has the property that a polynomial $f$ is in I if and only if it reduces to $0$.

Buchberger shows that the order in which reductions are applied is immaterial in the case of a Gröbner Basis. He also supplies some equivalent characterizations of a Gröbner Basis among which is a simple algorithmic test for a basis to have this property. The Gröbner Basis Algorithm is important because it provides an algorithmic method for determining ideal membership, and for determining a canonical simple representative for congruence classes mod an ideal.

## B. Operator Expressions.

We regard $x$, $y$, $x^{-1}$, $y^{-1}$, $(1-x)^{-1}$, etc. which arise in matrix expressions as atomic symbols—the variables in a ring of polynomials in non-commuting variables over the rational numbers. We let $F$ be a set of relations known to hold among these symbols—the relations are expressed as polynomials in the atomic variables which are known to be zero. We have, for example, the relations $xx^{-1}-1$ and $x^{-1}x-1$ which are "defining relations" (they define what is meant by the symbol $x^{-1}$ to be the inverse of the symbol $x$). Any member of the ideal generated by $F$ is clearly a relation on the atomic symbols. The reduction process described above can be carried over to this situation (provided that care is taken to respect the non-commutative nature of multiplication).

### 1. The Reduction Process.

Let $F = \{f_1, \ldots, f_k\}$ be a set of polynomials. Using the given ordering, each can be written with the monomials in descending order. Now let $f$ be any polynomial. We say that $f$ is reducible to $g$ with respect to $F$ if $g = f - cuf_iv$ where $c$ is a constant; $u$, $v$ are monomials chosen so that the leading term of $cuf_iv$ coincides with one of the terms of $f$. The effect is to replace a term of $f$ with terms of lower order.

A reduction step can be conceived of as a replacement LHS$\longrightarrow$RHS of a term in $f$ containing LHS as a factor by a term or sum of terms of lower order. If, for example, a term of $f$ contains $x\ x^{-1}$, then it can be replaced by $1$. It should be noted that this process is like the method that analysts use when simplifying operator expressions "by hand"; complicated subexpressions are repeatedly replaced by simpler ones.

The description of the reduction procedure in terms of replacement rules corresponds to the way it is commonly implemented. The "handedness" of replacement rules is determined by the term ordering. The LHS is always taken to be the leading term of a polynomial relation, while RHS is the negative of the sum of the remaining terms (which are of lower order).

**2. The Basis Algorithm.**

The reduction process is simpler than and independent of the Basis Algorithm which generates a set of relations "once and for all" for each set of atomic expressions. The generation of relations can be quite time-consuming. Once the relations are at hand, reduction of expressions is relatively fast. (The Basis Algorithm algorithm is part of a Forth-based system designed by Wavrik for the study of reduction processes of this type. The reduction process implementing some of the (NF) relations is available as part of a Mathematica linear algebra package [NFA].)

As in the case of commuting variables, it cannot be expected that an arbitrary set of relations provides a "good" set of replacement rules. Our aim is to apply a non-commutative version of the Gröbner Basis Algorithm to obtain a set of replacement rules which allow the simplification process to be applied mechanically.

As in the commutative case, a Gröbner Basis, $G$, for an ideal, I, is a set of polynomials in I having the property that a polynomial $f$ is in I if and only if N-Form$(f, G) = 0$. In the commutative case a finite Gröbner Basis exists for any ideal and can be obtained by the Buchberger Algorithm applied to any system of generators for the idea. For ideals in a non-commutative polynomial ring it need not be true that an ideal has a finite Gröbner Basis. The following algorithm, due to $F$. Mora [Mora], is an adaptation of Buchberger's Algorithm to the case of non-commutative polynomial rings.

Let $S$ be the free semigroup generated by a finite alphabet $A$. Let $(m_1, m_2)$ be an ordered pair of elements of $S$. By a match of $(m_1, m_2)$ we mean a 4-tuple $(l_1, r_1, l_2, r_2)$ of elements of $S$ which satisfy one of the following conditions:

(1) $l_1 = r_1 = 1$, $m_1 = l_2 m_2 r_2$.

(2) $l_2 = r_2 = 1$, $m_2 = l_1 m_1 r_1$.

(3) $l_1 = r_2 = 1$, $l_2 \neq 1$, $r_1 \neq 1$, there is a $w \neq 1$ with $m_1 = l_2 w$, $m_2 = w r_1$.

(4) $l_2 = r_1 = 1$, $l_1 \neq 1$, $r_2 \neq 1$, there is a $w \neq 1$ with $m_1 = w r_2$, $m_2 = l_1 w$.

These conditions make $l_1 m_1 r_1 = l_2 m_2 r_2$ a common multiple of $m_1$ and $m_2$ which is minimal in some sense.

If $f$ is a polynomial, let $lt(f)$ denote the leading term of $f$ (with respect to the given ordering). $lt(f) = lc(f)\ lm(f)$ where $lc(f)$ is the leading coefficient and $lm(f)$ the leading monomial. In the commutative case, the Basis Algorithm makes use of a kind of resolvent of two polynomials called the S-Polynomial. S-Pol$(f,g) = c_2 u_1 f_1 - c_1 u_2 f_2$ where $c_i = lc(f_i)$ and where $u_i$ is chosen so that $u_i lm(f_i)$ is the least common multiple of $lm(f_1)$ and $lm(f_2)$. In the non-commutative case, there are several such resolvents—one for each match. We denote by $M$ a 6-tuple $(f_1, f_2, l_1, r_1, l_2, r_2)$ where $(l_1, r_1, l_2, r_2)$ is a match for $(lm(f_1), lm(f_2))$ and $c_i = lc(f_i)$. Let

$$\text{S-Pol}(M) = c_2 l_1 f_1 r_1 - c_1 l_2 f_2 r_2 \ .$$

**Example**:

Consider the polynomials

$$\begin{aligned} f_1 &= aaba + ab \\ f_2 &= abba + ba \end{aligned} .$$

There are four matches for $(f_1, f_2)$:

1. $(aba, 1, 1, aba)$. In this case the S-Polynomial is

$$(aba)(f_1) - (f_2)(aba) = -baaba + abaab \ .$$

2. $(ab, 1, 1, ba)$ In this case the S-polynomial is

$$(ab)(f_1) - (f_2)(ba) = -baba + abab \ .$$

3. $(1, baa, aab, 1)$ In this case the S-Polynomial is

$$(f_1)(baa) - (aab)(f_2) = abbaa - aabba \ .$$

4. $(1, a, a, 1)$ In this case the S-Polynomial is

$$(f_1)(a) - (a)(f_2) = 0 \ .$$

The algorithm takes the given basis and adds to it all nonzero reductions of S-Polynomials for all pairs in the original basis. The process repeats as long as there are S-Polynomials with nonzero reduction.

**MORA'S BASIS ALGORITHM** (MBA):

Input: A set $F$ of polynomials
Output: A Gröbner basis, $G$, for the ideal generated by $F$
(if the algorithm terminates)

```
n := 1, H_1 := F, G_1 := F
While not-empty(H_n) do
    set-empty(B_n), set-empty(H_{n+1})
    For every match M = (f_1, f_2, l_1, r_1, l_2, r_2)
        with f_1 in G_n, f_2 in H_n do
        adjoin(B_n, M)
    While not-empty(B_n) do
        extract M from B_n
        f := S-Pol(M)
        f := N-Form(f, union(G_n, H_{n+1}))
        If f ≠ 0 then adjoin(H_{n+1}, f)
    G_{n+1} := union(G_n, H_{n+1})
    n := n + 1
G := G_n
```

In the current work, a final reduction step is used:

```
Repeat
        unchanged := true
        set-empty(G_1)
        While not-empty(G) do
                extract f from G
                f_1 :=N-Form(f, union(G, G_1))
                If f_1 ≠ f then unchanged := false
                If f_1 ≠ 0 then adjoin(G_1, f_1)
        G := G_1
Until unchanged
```

Mora shows that, if the process terminates, $G$ is a finite Gröbner Basis for the ideal generated by $F$.

**C. Term Ordering**

The term ordering used in this paper is the graded lexicographic order. Let $M_1$ and $M_2$ be monomials in the atomic symbols. We define $M_1 < M_2$ if either

$$\text{length } (M_1) < \text{length } (M_2)$$

or length $(M_1) =$ length $(M_2)$ and $M_1$ comes before $M_2$ in lexicographic order. This ordering depends on the choice of ordering (say lexicographic) on the atomic symbols.

**Example.** If $a < b < c$ then the monomials of length $3$ are ordered

$$\begin{aligned}&aaa < aab < aac < aba < abb < abc < aca < acb < acc < baa < bab <\\ &bac < bba < bbb < bbc < bca < bcb < bcc < caa < cab < cac < cba <\\ &cbb < cbc < cca < ccb < ccc\end{aligned}$$

The concept of simplification by use of replacement rules involves an underlying ordering on the terms of expressions. For the cases we study there are many natural ways to put an ordering on terms. We select one which reflects our notion of going from complicated to less complicated. While other orderings are possible, there are advantages to the user if a theoretically "simplified" expression looks simple. Here is the one we use for EB: The intuition we capture in our ordering is:

square roots of complicated polynomials = most complicated.

square roots of simple polynomials = second most complicated.

inv of complicated polynomials = third complicated.

inv of simple polynomials = next most complicated.

complicated monomials

simple monomials

commuting elements and expressions in them.

This is merely a partial order rather than a total order so in addition we specify lexicographic order on the underlying variables.

In other notation the ordering on EB is

$$\begin{matrix} x \\ y \end{matrix} < \begin{matrix} x^{-1} \\ y^{-1} \end{matrix} < \begin{matrix} (1-xy)^{-1} \\ (1-yx)^{-1} \end{matrix}$$

It is a partial order, but both in computing the basis and in applying the reduction process it is made a total ordering by imposing the lexicographic order $x < y$.

In this ordering we will always write rules so that they move complicated expressions to the right. The effect is that any replacement rule will replace expressions of high order by those of lower order. (A set of replacement rules which is not consistent in this way can produce infinite loops of replacements.)

For example, the identity $(1-xy)^{-1}\,x = x(1-yx)^{-1}$ can be converted to a replacement rule in two ways. The one consistent with the above order is:

$$(1-xy)^{-1}x \longrightarrow x(1-yx)^{-1}$$

in that the complicated inv moves right. The wrong rule is

$$x(1-yx)^{-1} \longrightarrow (1-xy)^{-1}x\ .$$

For the NF case we use

$$\begin{aligned} x < y < x^{-1} < y^{-1} < (1-x)^{-1} < (1-y)^{-1} < (1-xy)^{-1} \\ < (1-yx)^{-1} < (1-xy)^{1/2} < (1-yx)^{1/2}\ . \end{aligned}$$

It should be noted that expressions involving higher powers of $(1-yx)^{1/2}$ and $(1-xy)^{1/2}$ are ordered in the standard way. However, among the replacement rules are

$$((1-yx)^{1/2})^2 \longrightarrow 1-yx$$

$$((1-xy)^{1/2})^2 \longrightarrow 1-xy$$

so any expression involving higher powers of $(\ )^{1/2}$ is reduced. All higher powers will be replaced by lower order expressions.

## III. OPERATOR RELATIONS WITH FINITE BASIS FOR RULES

### A. The (EB) and (RESOL) Relations

The (RESOL) case uses graded lexicographic ordering induced by the following order of the variables:

$$x < x^{-1} < (1-x)^{-1}$$

while for the (EB) case we use:

$$x < y < x^{-1} < y^{-1} < (1-x)^{-1} < (1-y)^{-1} \ .$$

Input to the algorithm are the defining relations for the inverses which are involved.

**Theorem III.1.** *The Mora Basis algorithm applied to* (EB) *and* (RESOL) *with the ordering above stops in finitely many steps and produces the relations below. The resulting set of relations is a finite Gröbner Basis.*

(RESOL)

Relations 0-3 are the starting relations, 4-5 were deduced by the algorithm.

$$\begin{aligned}
\mathrm{Res}_0 &= x^{-1}x - 1 \\
\mathrm{Res}_1 &= xx^{-1} - 1 \\
\mathrm{Res}_2 &= (1-x)^{-1}\,x - (1-x)^{-1} + 1 \\
\mathrm{Res}_3 &= x\,(1-x)^{-1} - (1-x)^{-1} + 1
\end{aligned}$$

$$\begin{aligned}
\mathrm{Res}_4 &= (1-x)^{-1}\,x^{-1} - (1-x)^{-1} - x^{-1} \\
\mathrm{Res}_5 &= x^{-1}\,(1-x)^{-1} - (1-x)^{-1} - x^{-1} \ .
\end{aligned}$$

(EB)

The starting relations for the process are the defining relations for the inverse function:

$$\mathrm{EB}_0 = x^{-1}\, x - 1$$

$$\mathrm{EB}_1 = x x^{-1} - 1$$

$$\mathrm{EB}_2 = y^{-1}\, y - 1$$

$$\mathrm{EB}_3 = \; y y^{-1} - 1$$

$$\mathrm{EB}_4 = xy\, (1 - xy)^{-1} - (1 - xy)^{-1} + 1$$

$$\mathrm{EB}_5 = yx\, (1 - yx)^{-1} - (1 - yx)^{-1} + 1$$

with

$$\mathrm{EB}_6 = (1 - xy)^{-1} xy - (1 - xy)^{-1} + 1$$

$$\mathrm{EB}_7 = (1 - yx)^{-1} yx - (1 - yx)^{-1} + 1$$

The Mora Basis Algorithm terminates and produces precisely the following additional relations

$$\mathrm{EB}_8 = (1 - yx)^{-1}\, x^{-1} - y(1 - xy)^{-1} - x^{-1}$$

$$\mathrm{EB}_9 = (1 - xy)^{-1}\, y^{-1} - x\, (1 - yx)^{-1} - y^{-1}$$

$$\mathrm{EB}_{10} = x^{-1}\, (1 - xy)^{-1} - y\, (1 - xy)^{-1} - x^{-1}$$

$$\mathrm{EB}_{11} = y^{-1}\, (1 - yx)^{-1} - x\, (1 - yx)^{-1} - y^{-1}$$

$$\mathrm{EB}_{12} = (1 - yx)^{-1}\, y - y\, (1 - xy)^{-1}$$

$$\mathrm{EB}_{13} = (1 - xy)^{-1}\, x - x\, (1 - yx)^{-1}$$

The termination of the algorithm implies that these form a Gröbner Basis for the ideal generated by the initial relations. When used as replacement rules they provide a canonical simplification for any expression that only involves these variables.

## IV. OPERATOR RELATIONS WITH INFINITE BASIS FOR RULES

### A. (preNF) and (NF) Relations

The ordering used is:

$$\text{(preNF)} \quad x < y < x^{-1} < y^{-1} < (1-x)^{-1} < (1-y)^{-1} < (1-xy)^{-1} < (1-yx)^{-1}$$

(NF) *above* $< (1-xy)^{1/2} < (1-yx)^{1/2}$ .

Notice that the (preNF) expressions equal (RESOL) and (EB) together with a particular ordering. In the preNF, NF and NF+ situations we find that the Mora Basis Algorithm does not terminate. It produces, instead, a small collection of special relations followed by an unending set of sequences of relations. These can be regarded as general relations. A typical relation of this type is

$$x\,(1-yx)^{-n}\,(1-x)^{-1} - (1-xy)^{-n}\,\{(1-x)^{-1}-1\}$$

expanded out. The relations listed below are valid for $n = 1, 2, \ldots$. Relations with a * are also valid for $n = 0$. The relations for each of these cases are based on the following set of 16 general relations.

$I^*$. $x(1-yx)^{-n}(1-x)^{-1} - (1-xy)^{-n}\{(1-x)^{-1}-1\}$

II. $x(1-yx)^{-n}(1-y)^{-1} - (1-xy)^{-n}(1-y)^{-1} - x(1-yx)^{-n} + (1-xy)^{-(n-1)}(1-y)^{-1}$

$III^*$. $x(1-yx)^{-n}(1-yx)^{1/2}(1-x)^{-1} - (1-xy)^{-n}(1-xy)^{1/2}\{(1-x)^{-1}-1\}$

IV. $x(1-yx)^{-n}(1-yx)^{1/2}(1-y)^{-1} - \{(1-xy)^{-n} - (1-xy)^{-(n-1)}\}(1-xy)^{1/2}(1-y)^{-1} - x(1-yx)^{-n}(1-yx)^{1/2}$

V. $y(1-xy)^{-n}(1-x)^{-1} - \{(1-yx)^{-n} - (1-yx)^{-(n-1)}\}(1-x)^{-1} - y(1-xy)^{-n}$

$VI^*$. $y(1-xy)^{-n}(1-y)^{-1} - (1-yx)^{-n}\{(1-y)^{-1}-1\}$

$VII^*$. $y(1-xy)^{-n}(1-xy)^{1/2}(1-x)^{-1} - \{(1-yx)^{-n} - (1-yx)^{-(n-1)}\}(1-yx)^{1/2}(1-x)^{-1} - y(1-xy)^{-n}(1-xy)^{1/2}$

$VIII^*$. $y(1-xy)^{-n}(1-xy)^{1/2}(1-y)^{-1} - (1-yx)^{-n}(1-yx)^{1/2}\{(1-y)^{-1}-1\}$

$IX^*$. $(1-x)^{-1}(1-yx)^{-n}(1-x)^{-1} - (1-x)^{-1}(1-xy)^{-n}(1-x)^{-1} - (1-yx)^{-n}(1-x)^{-1} + (1-x)^{-1}(1-xy)^{-n}$

X. $(1-x)^{-1}(1-yx)^{-n}(1-y)^{-1} - (1-x)^{-1}\{(1-xy)^{-n} - (1-xy)^{-(n-1)}\}(1-y)^{-1} - (1-yx)^{-n}\{(1-y)^{-1}-1\} - (1-x)^{-1}(1-yx)^{-n}$

XI. $(1-x)^{-1}(1-yx)^{-n}(1-yx)^{1/2}(1-x)^{-1} - (1-x)^{-1}(1-xy)^{-n}(1-xy)^{1/2}\{(1-x)^{-1} - 1\} - (1-yx)^{-n}(1-yx)^{1/2}(1-x)^{-1}$

XII. $(1-x)^{-1}(1-yx)^{-n}(1-yx)^{1/2}(1-y)^{-1}-(1-x)^{-1}\{(1-xy)^{-n}-(1-xy)^{-(n-1)}\}$ $(1-xy)^{1/2}(1-y)^{-1}-(1-yx)^{-n}(1-yx)^{1/2}\{(1-y)^{-1}-1\}-(1-x)^{-1}(1-yx)^{-n}(1-yx)^{1/2}$

XIII. $(1-y)^{-1}(1-yx)^{-n}(1-x)^{-1}-(1-y)^{-1}\{(1-xy)^{-n}+\ldots+1\}(1-x)^{-1}+$ $\{(1-xy)^{-n}+\ldots+(1-xy)^{-1}\}(1-x)^{-1}+(1-y)^{-1}\{(1-xy)^{-n}+\ldots+(1-xy)^{-1}\}-$ $\{(1-xy)^{-n}+\ldots+(1-xy)^{-1}\}$

$XIV^*$. $(1-y)^{-1}(1-yx)^{-n}(1-y)^{-1}-(1-y)^{-1}(1-xy)^{-n}(1-y)^{-1}+(1-xy)^{-n}(1-y)^{-1}-$ $(1-y)^{-1}(1-yx)^{-n}$

XV. $(1-y)^{-1}(1-yx)^{-n}(1-yx)^{1/2}(1-x)^{-1}-(1-y)^{-1}\{(1-xy)^{-n}+\ldots+(1-xy)^{-1}\}$ $(1-xy)^{1/2}(1-x)^{-1}+\{1-xy)^{-n}+\ldots+(1-xy)^{-1}\}1-xy)^{1/2}(1-x)^{-1}+(1-y)^{-1}$ $\{1-xy)^{-n}+\ldots+(1-xy)^{-1}\}(1-xy)^{1/2}-\{(1-xy)^{-n}+\ldots+(1-xy)^{-1}\}(1-xy)^{1/2}-$ $(1-y)^{-1}(1-yx)^{1/2}(1-x)^{-1}$

$XVI^*$. $(1-y)^{-1}(1-yx)^{-n}(1-yx)^{1/2}(1-y)^{-1}-(1-y)^{-1}(1-xy)^{-n}(1-xy)^{1/2}(1-y)^{-1}+$ $(1-xy)^{-n}(1-xy)^{1/2}(1-y)^{-1}-(1-y)^{-1}(1-yx)^{-n}(1-yx)^{1/2}$

The rules can be paired. In the left column below are the relations for the preNF case which do not involve $[\ ]^{1/2}$. The corresponding right member is obtained from the left by inserting a $[\ ]^{1/2}$ factor.

| | |
|---|---|
| $I$ | $III$ |
| $II$ | $IV$ |
| $V$ | $VII$ |
| $VI$ | $VIII$ |
| $IX$ | $XI$ |
| $X$ | $XII$ |
| $XIII$ | $XV$ |
| $XIV$ | $XVI$ |

The transformation of the relations on the left to those on the right is (with the exception of $XIII \longrightarrow XV$) to replace

$$(1-xy)^{-n} \quad \text{by} \quad (1-xy)^{-n}(1-xy)^{1/2}$$

$$(1-yx)^{-n} \quad \text{by} \quad (1-yx)^{-n}(1-yx)^{1/2}$$

While the set of relations produced by the Gröbner Basis Algorithm is infinite in the polynomial setting, in Section V we will discuss below a general algebra, $A_{\mathcal{F}}$, in which the infinite sequences can be considered as members of a finite set.

**1.** (preNF)

The starting relations used in the preNF case are the defining relations for $z^{-1}$ where $z$ is $x$, $y$, $1-x$, $1-y$, $1-xy$, and $1-yx$. Note that these are the same as the starting relations for (EB) and (RESOL) on $x$ and on $y$.

**Theorem IV.1.** *The Mora Basis algorithm for* (preNF) *does not terminate but ultimately generates relations contained in the eight classes listed in the left column above (viz* I, II, V, VI, IX, X, XIII, XIV*). In addition to the starting relations, the algorithm produces the relations from* (EB) *and* (RESOL) *listed earlier. It also produces the following particular relations which do not fit into the eight general classes.*

$$\mathrm{rel[Pre1]} = (1-x)^{-1}y(1-xy)^{-1} - (1-x)^{-1}(1-xy)^{-1} - y(1-xy)^{-1} + (1-x)^{-1}$$

$$\mathrm{rel[Pre2]} = (1-y)^{-1}x(1-yx)^{-1} - (1-y)^{-1}(1-yx)^{-1} - x(1-yx)^{-1} + (1-y)^{-1}$$

*All of these relations are algebraic consequences of the starting relations (which are just the defining relations for inverse). They are obtained from the starting relations strictly by the application of polynomial arithmetic.*

**2.** (NF)

We add the symbols $(1-xy)^{1/2}$ and $(1-yx)^{1/2}$ to the (preNF) setting and add the relations

$$\mathrm{NFDEF}_1 = ((1-xy)^{1/2})^2 + xy - 1$$

$$\mathrm{NFDEF}_2 = ((1-yx)^{1/2})^2 + yx - 1$$

(defining relations for $(\ )^{1/2}$) to the starting relations. We have seen that the Mora process for (preNF) produces

$$(1-xy)^{-1}x - x(1-yx)^{-1}$$

$$(1-yx)^{-1}y - y(1-xy)^{-1}$$

These yield important simplifications but appear relatively late in the process. They are, in the interest of efficiency, added to the starting relations for (NF).

Companion relations for $(\ )^{1/2}$

$$(1-xy)^{1/2}x - x(1-yx)^{1/2}$$

$$(1-yx)^{1/2}y - y(1-xy)^{1/2}$$

These relations will not arise from applying the Basis Algorithm to the defining relations. We have therefore added them to the starting relations in the (NF) case. These can be proved by an analytic argument. These relations are extremely important to those working with the Nagy-Foias model.

**Theorem IV.2.** *The Mora Basis algorithm for* (NF) *does not terminate but ultimately generates the relations from* (preNF) *together with the remaining eight classes from the* 16 *above (viz* III, IV, VII, VIII, XI, XII, XV, XVI*). It also produces the following particular relations which do not fit into the* 16 *general classes.*

$$\mathrm{rel[NF1]} = (1-xy)^{1/2}y^{-1} - y^{-1}(1-yx)^{1/2}$$

$$\mathrm{rel[NF2]} = (1-xy)^{1/2}(1-xy)^{-1} - (1-xy)^{-1}(1-xy)^{1/2}$$

$$\mathrm{rel[NF3]} = (1-yx)^{1/2}x^{-1} - x^{-1}(1-xy)^{1/2}$$

$$\mathrm{rel[NF4]} = (1-yx)^{1/2}(1-yx)^{-1} - (1-yx)^{-1}(1-yx)^{1/2}$$

$$\mathrm{rel[NF5]} = x^{-1}(1-xy)^{1/2}(1-x)^{-1} - (1-yx)^{1/2}(1-x)^{-1} - x^{-1}(1-xy)^{1/2}$$

$$\mathrm{rel[NF6]} = y^{-1}(1-yx)^{1/2}(1-y)^{-1} - (1-xy)^{1/2}(1-y)^{-1} - y^{-1}(1-yx)^{1/2}$$

## B. A Note on Symmetry

In terms of operator theory, nothing distinguishes $x$ from $y$. The Simplification Process, however, requires a term ordering—and we have imposed an ordering in which $y$ has a higher precedence than $x$. We examine the extent to which the reduction rules are symmetric in $x$ and $y$. The starting relations for these classes are symmetric if one interchanges $x$ and $y$, but the Basis Algorithm depends on the term ordering and reduction rules which are not entirely symmetric. For reference below we denote by $xchg(r)$ the result of making the exchange of $x$ with $y$ in relation $r$. The change also takes place within atomic symbols—thus $xchg((1-x)^{-1}) = (1-y)^{-1}$ and $xchg((1-xy)^{-1}) = (1-yx)^{-1}$.

Here are the rules related by $xchg$:

$$\begin{aligned} \mathrm{I} &= xchg(\mathrm{VI}) & \qquad \mathrm{III} &= xchg(\mathrm{VIII}) \\ \mathrm{II} &= xchg(\mathrm{V}) & \qquad \mathrm{IV} &= xchg(\mathrm{VII}) \\ \mathrm{IX} &= xchg(\mathrm{XIV}) & \qquad \mathrm{XI} &= xchg(\mathrm{XVI}) \end{aligned}$$

The classes X and XIII (also XII and XV) are not directly related by $xchg$ but satisfy the following inductive formulas:

$$\begin{aligned} \mathrm{XIII}(n) &= -xchg(\mathrm{X}(n)) + \mathrm{XIII}(n-1) \\ \mathrm{XV}(n) &= -xchg(\mathrm{XII}(n)) + \mathrm{XV}(n-1) \end{aligned}$$

## V. A NEW ALGEBRA CONTAINING THE FUNCTIONAL CALCULUS OF OPERATOR THEORY

### A. The General Relations

To avoid dazzling with abstraction we begin by stating some explicit rules which contain one half of the rules in section IV as special cases. These seem like basic simplification rules for much of operator theory. Next we show a setting natural for treating such rules and obtaining Gröbner basis properties.

(GENR)

GR0 $\quad h(xy)x \longrightarrow xh(yx)$.

GR1 $\quad h(yx)x^{-1} \longrightarrow x^{-1}h(xy)$.

GR2 $\quad xh(yx)(1-x)^{-1} \longrightarrow -h(xy) + h(xy)(1-x)^{-1}$.

GR3 $\quad x^{-1}h(xy)(1-x)^{-1} \longrightarrow x^{-1}h(xy) + h(yx)(1-x)^{-1}$.

GR4 $\quad (1-x)^{-1}h(yx)(1-x)^{-1} - (1-x)^{-1}h(xy)(1-x)^{-1}$

$\qquad \longrightarrow h(yx)(1-x)^{-1} - (1-x)^{-1}h(xy)$.

for all operators $x, y$ on a Hilbert space H with $x$, $1-x$ invertible and functions $h$ analytic on the spectrum of $xy$ and $yx$.

Operators of the form $h(xy)$ are of course what is called the *functional calculus* of the operator $xy$. Consequently we are preparing to treat general rules which are true for general functional calculi; one of the most important constructions in operator theory.

There are several observations about (GENR). First while (GR4) is not in the same form as the reduction rules considered previously, it transforms a sum rather than a monomial. To build a conventional rule from this one must use lexicographic order. The second observation is that (GENR) are easy to prove directly without reference to any algorithm for their construction.

**Proof.** (GENR) The proof is easy. A typical example is

$$\text{(GR3)}\ x^{-1}h(xy)(1-x)^{-1} = h(yx)x^{-1}(1-x)^{-1} =$$
$$= h(yx)(x^{-1}+(1-x)^{-1}) = h(yx)x^{-1} + h(yx)(1-x)^{-1}$$

which yields (GR3). The other cases are similar.

The main point about these rules is that they contain many of the preNF and NF rules as a special case. Indeed

**THEOREM V.1.** *Rules* I, III, VI, VIII, IX, XI, XIV, XVI *of section* 4 *follow from* (GENR) *provided $xy$ and $yx$ have spectrum inside the open unit disk.*

**Proof.** The proof consists of comparing the list in section IV against (GENR). For example rule III is obtained from (GR2) by setting $h[s] = (1-s)^{-n}(1-s)^{1/2}$ for $s$ in the unit disk in the complex plane $\mathbf{C}$. Since the spectrum of $xy$ is in the unit disk and $h$ is analytic there (GR2) applies to give rule III of section IV.

**B. The Functional Calculus Algebra**

What is missing in our study so far is a unified way of dealing with infinite sets of rules. We now introduce a setting which does this for the situation here and may be good for doing lots of the algebra which occurs in functional analysis.

Let $\mathcal{F}$ denote the set of analytic functions

$$f : \mathcal{D} \subset \mathbf{C} \longrightarrow \mathbf{C}\ ,$$

on an open domain $\mathcal{D}$. Given $A$ the algebra of all bounded operators on $H$ define the *functional calculus algebra* $A_{\mathcal{F}}$ for $A$ to be the set of $A$ valued functions on $\mathcal{F}$. It is an algebra with the multiplication on

$$A_{\mathcal{F}} = \{g : \mathcal{F} \longrightarrow A\}$$

defined by $uv[h] = u[h]v[h]$. Note the constant functions in $A_{\mathcal{F}}$ can be identified with $A$ as follows: Denote by $c_v$ the element $c$ in $A_{\mathcal{F}}$ defined by $c[h] = v$ where $v$ is a constant function. Then $c_a c_b = c_{ab}$.

We organize our approach differently than before to take advantage of the power of this more general framework.

We now define certain elements in the algebra $A_{\mathcal{F}}$ and give the defining relations for them so that when we apply the (MBA) algorithm we get (GENR).

**C.** RESOL **Plus Generalized** EB

Fix $x, y$ that is the constant functions $c_x$ $c_y$ in the algebra $A_{\mathcal{F}}$. Then $x^{-1}$, $y^{-1}$, $(1-x)^{-1}$, $(1-y)^{-1}$ correspond to $c_{x^{-1}}$, $c_{y^{-1}}$, $c_{(1-x)^{-1}}$, $c_{(1-y)^{-1}}$ in the algebra $A_{\mathcal{F}}$ and indeed satisfy the same relations, namely, the (RESOL) relations.

Fix two functions $S, T$ in $A_{\mathcal{F}}$ defined by

$$S(h) = h(xy) \quad T(h) = h(yx)$$

These satisfy

$$\begin{aligned} S\ c_x &= c_x\ T \quad \text{since} \quad h(xy)x = xh(yx) \\ c_y\ S &= T\ c_y \quad \text{since} \quad yh(xy) = h(yx)y \end{aligned}.$$

This somewhat complicated setup has the following rather simple algebraic structure which is what we use when implementing (MBA) on a computer.

We denote by $\mathcal{R}_1^{\mathcal{F}}$ the elements $c_x$, $c_y$, $c_{x^{-1}}$, $c_{y^{-1}}$, $c_{(1-x)^{-1}}$, $c_{(1-y)^{-1}}$, $S$ and $T$ of $A_{\mathcal{F}}$ together with the ordering on formal monomials in them based on degree of the monomial plus

$$\begin{matrix} c_x \\ c_y \end{matrix} < \begin{matrix} c_{x^{-1}} \\ c_{y^{-1}} \end{matrix} < \begin{matrix} c_{(1-x)^{-1}} \\ c_{(1-y)^{-1}} \end{matrix} < \begin{matrix} S \\ T \end{matrix}$$

Instead of working directly with $\mathcal{R}_1^{\mathcal{F}}$, a subset of $A_{\mathcal{F}}$, we introduce an abstraction of the algebra involved.

**DEFINITION**: Let $\mathcal{R}_1$ denote the polynomial ring generated by the atomic symbols used in (RESOL)

$$\begin{matrix} x,\ (1-x)^{-1},\ x^{-1} \\ y,\ (1-y)^{-1},\ y^{-1} \end{matrix}$$

plus two additional elements $S$ and $T$ which satisfy (5.1)

$$\begin{aligned} Sx &= xT \\ yS &= Ty \end{aligned}$$

The ordering on monomials is the usual plus

$$\begin{matrix} x \\ y \end{matrix} < \begin{matrix} x^{-1} \\ y^{-1} \end{matrix} < \begin{matrix} (1-x)^{-1} \\ (1-y)^{-1} \end{matrix} < \begin{matrix} S \\ T \end{matrix}$$

The relations we use in $\mathcal{R}_1$ are satisfied by the corresponding elements of $\mathcal{R}_1^{\mathcal{F}}$ considered as elements of $A_{\mathcal{F}}$. Let $\mathcal{I}_1$ be the ideal in $\mathcal{R}_1$ generated by the starting relations. $\mathcal{I}_1$ is invariant under the substitution sending $S$ to $S^n$ and $T$ to $T^n$. This is true of the starting generators hence of any element in the ideal. Our main result shows that (GENR) is a type of Gröbner basis produced for $\mathcal{R}_1^{\mathcal{F}}$. We now state it formally, but with the obvious abbreviated notation $c_v$ replaced by $v$.

**THEOREM V.2.** *The* (RESOL) *relations and the list of relations:*

(GA0) $$Sx - xT \qquad yS - Ty$$

*plus*

(GA1) $$x^{-1}S - Tx^{-1} \quad Sy^{-1} - y^{-1}T$$

(GA2) $$\begin{aligned} & xT(1-x)^{-1} - S(1-x)^{-1} - S \\ & yS(1-y)^{-1} - T(1-y)^{-1} + T \end{aligned}$$

(GA3) $$\begin{aligned} & x^{-1}S(1-x)^{-1} - T(1-x)^{-1} - x^{-1}S \\ & y^{-1}T(1-y)^{-1} - S(1-y)^{-1} - y^{-1}T \end{aligned}$$

(GA4) $$\begin{aligned} & (1-x)^{-1}T(1-x)^{-1} - (1-x)^{-1}S(1-x)^{-1} - T(1-x)^{-1} + (1-x)^{-1}S \\ & (1-y)^{-1}T(1-y)^{-1} - (1-y)^{-1}S(1-y)^{-1} + S(1-y)^{-1} - (1-y)^{-1}T \end{aligned}$$

*form a Gröbner basis for the ideal they generate (inside $\mathcal{R}_1$) provided we use the property imposed by $\mathcal{R}_1^{\mathcal{F}}$ that if a relation contains $S$, $T$ it is also true for $S$, $T$ replaced by $S^n$, $T^n$. Note these relations are produced by the (MBA).*

The proof is given in section VI where we also give some basics about Gröbner bases.

**D. The preNF Relations**

The previous section concerned relations involving a general analytic function $h$. If one restricts $h$ to be in a special class of functions, then it is likely that more rules are true and that one gets a set of rules which contain (GR0–4). For example if $h(s) = (1-s)^{-n}$, then (with the previous choice of ordering) one gets the preNF rules and these contain rules in addition to those arising from (RES) alone, (EB) alone, and (GR0–4) In this section we show how treating addition of the special structure $h(s) = (1-s)^{-n}$ and $h(s) = (1-s)^{-n}(1-s)^{-1/2}$ can be put in the context of the functional calculus algebra. Thereby we can treat problems of varying levels of generality.

We start with

**DEFINITION**: $\mathcal{R}_2$ is the algebra $\mathcal{R}_1$ plus $U, V$ defined by

$$\begin{aligned} U &= (1-xy)S \\ V &= (1-yx)T \end{aligned}$$

$Ux = xV \;\; yU = Vy$.

Now we set the ordering so that

$$\text{all the rest} \;\; < \begin{matrix} U \\ V \end{matrix} < \begin{matrix} S \\ T \end{matrix} .$$

This is counter intuitive because $U = (1-xy)S$ seems formally to have higher degree than $S$.

We motivate this construction by pointing out that it naturally sits in $A_{\mathcal{F}}$ where one defines $U$ and $V$ by

$$U(h) = (1-xy)S(h)$$

$$V(h) = (1-yx)T(h)$$

and to motivate the ordering think of $h$ as having an expansion

$$h(s) = \sum_{n=0}^{\infty} a_n (1-s)^{-n} .$$

Then $U(h) = (1-xy)S(h)$ because of cancellation has lower degree than $S(h) = h(xy)$.

What we find upon running Mora's algorithm are relations which can be summarized by:

Those gotten from (RESOL) and(EB) plus the two special relations rel[Pre1] and rel[Pre2]. Plus

GENREL[1] $= xT^nV^m(1-x)^{-1} - S^nU^m(1-x)^{-1} + S^nU^m$

GENREL[2]
$= xT^nV^m(1-y)^{-1} + S^{n-1}U^{m+1}(1-y)^{-1} - S^nU^m(1-y)^{-1} - xT^nV^m$

GENREL[3] $= yS^nU^m(1-x)^{-1} + T^{n-1}V^{m+1}(1-x)^{-1} - T^nV^m(1-x)^{-1} - yS^nU^m$

GENREL[4] $= yS^nU^m(1-y)^{-1} - T^nV^m(1-y)^{-1} + T^nV^m$

GENREL[5] $= x^{-1}S^n(1-x)^{-1} - T^n(1-x)^{-1} - x^{-1}S^n$

GENREL[6] $= x^{-1}S^nU - x^{-1}S^{n+1} + yS^{n+1}$

GENREL[7] $= y^{-1}T^n(1-y)^{-1} - S^n(1-y)^{-1} - y^{-1}T^n$

GENREL[8] $= y^{-1}T^nV - y^{-1}T^{n+1} + xT^{n+1}$

GENREL[9] $= (1-x)^{-1}T^nV^m(1-x)^{-1} - (1-x)^{-1}S^nU^m(1-x)^{-1} -$
$T^nV^m(1-x)^{-1} + (1-x)^{-1}S^nU^m$

GENREL[10] $=$
$(1-x)^{-1}T^nV^m(1-y)^{-1} + (1-x)^{-1}S^{n-1}U^{m+1}(1-y)^{-1} -$
$(1-x)^{-1}S^nU^m(1-y)^{-1} - T^nV^m(1-y)^{-1} - (1-x)^{-1}T^nV^m + T^nV^m$

GENREL[11] $=$
$(1-x)^{-1}U^n(1-y)^{-1} + (1-x)^{-1}TV^{n-1}(1-y)^{-1} -$
$(1-x)^{-1}SU^{n-1}(1-y)^{-1} - TV^{n-1}(1-y)^{-1} - (1-x)^{-1}TV^{n-1} + TV^{n-1}$

GENREL[12] $=$
$(1-y)^{-1}T^nV^m(1-x)^{-1} - (1-y)^{-1}T^{n+m}(1-x)^{-1} + (1-y)^{-1}\sigma(1-x)^{-1}$
$- \sigma(1-x)^{-1} - (1-y)^{-1}\sigma + \sigma$

where $\sigma = \sum_{i=0}^{m-1} S^{m+n-i}U^i$ for $i = 0 \ldots m-1$.

GENREL[13] $=$
$(1-y)^{-1}T^nV^m(1-y)^{-1} - (1-y)^{-1}S^nU^m(1-y)^{-1} + S^nU^m(1-y)^{-1} -$
$(1-y)^{-1}T^nV^m$

We may use these to deduce some of the general relations in section IV. To see this observe that the elements $S$, $T$, $U$, $V$ of $A_{\mathcal{F}}$ satisfy GENREL[1–13] whenever they are applied to functions $h$ of the form $h(s) = (1-s)^{-k}$. Now we use that

$$S^m(h) = (1-xy)^{-mk} \qquad T^m(h) = (1-yx)^{-mk}$$
$$U^m(h) = (1-xy)^{-(k-1)m} \quad V^m(h) = (1-yx)^{-(k-1)m}$$

Thus the common expressions $T^nV^m$ and $S^nU^m$ become

$$T^nV^m \longrightarrow (1-yx)^{-nk-(k-1)m}$$
$$S^nU^m \longrightarrow (1-xy)^{-nk-(k-1)m}$$

Now we substitute these into GENREL[1–13] to reach the conclusion. As an example we treat GENREL[2]. When applied to $h(s) = (1-s)^{-k}$ it becomes

$$(1-xy)^{-r}(1-y)^{-1} - (1-xy)^{-(r-1)}(1-y)^{-1} + (1-xy)^{-r} - (1-yx)^{-r} \qquad \text{(IV.1)}$$

where we set $r = nk + (k-1)n$ and observe that $(n-1)k + (k-1)(m+l) = r-1$. Now (IV.1) equals II of section IV. Similar calculations yield preNF relations according to the table:

| GENREL | preNF |
|---|---|
| 1 | I |
| 2 | II |
| 3 | V |
| 4 | VI |
| 9 | IX |
| 10 | X |
| 13 | XIV |

This produces seven out of eight preNF relations; only XIII is missing. GENREL 5-8 and 12 after substituting $(1-xy)^{-1}$ become rules which are reduced to 0 by the rules already produced. GENREL 12 gives a relation which when subjected to induction gives XIII.

Consequently, we have an alternative way of producing the preNF relations to that given in section IV. Clearly it amounts to feeding different starting relations and ordering into the (MBA).

**E. The NF Relations in the $A_{\mathcal{F}}$ Setting**

We simply give the starting relations and ordering which seem appropriate to NF in the $A_{\mathcal{F}}$ setting.

**DEFINITION**: $\mathcal{R}_3$ is the algebra obtained by adjoining two elements $Q$, $R$ to $\mathcal{R}_2$.

$$Q^2 - (1-xy) = 0$$
$$R^2 - (1-yx) = 0$$

and

$$Qx - xR = 0$$
$$yQ - Ry = 0 \ .$$

(Intuitively $Q$ and $R$ behave like $(1-xy)^{1/2}$ and $(1-yx)^{1/2}$.) The ordering reflects

$$x < x^{-1} < (1-x)^{-1} < (1-xy)^{-1} < (1-xy)^{1/2} < (1-xy)^{-1/2}$$

and is

$$x < x^{-1} < (1-x)^{-1} < U < S < Q < Q^{-1}$$

with a similar ordering on $T$, $V$, $R$.

We have not tested this thoroughly but expect it will imply the NF

## VI. Gröbner BASIS PROPERTY

Let $I$ be a two-sided ideal in a polynomial ring, let $I^*$ denote the invertible elements in $I$, and let $G$ be a (possibly infinite) set of elements in $I^*$.

**DEFINITION** We say that a polynomial $f$ has a finite d-representation in terms of $G$ if there is a finite set of $a_i$ in $K$, $l_i$, $r_i$ monomials, $g_i$ in $G$ for $i = 1, \ldots, k$ so that $f = a_1 l_1 g_1 r_1 + \cdots + a_k l_k g_k r_k$ and where $lm(f) = lm(l_1 g_1 r_1) > \cdots > lm(l_k g_k r_k)$.

**THEOREM V.3.** *The following two conditions are equivalent:*

(1) *$f$ is in $I^*$ if and only if $f$ reduces to $0$ with respect to $G$.*

(2) *For every match $M = (g_1, g_2, l_1, r_1, l_2, r_2)$ with $g_1$, $g_2$ in $G$, S-Pol($M$) has a finite $d$-representation in terms of $G$.*

**Proof**: See [Mora] Theorem 3.3

$G$ is a Gröbner Basis if these equivalent conditions hold. For Gröbner bases the reduction process is well-behaved. Notice that condition (1) tells us that $N$-form$(f, G)$ gives a canonical representative for the equivalence class of $f$ mod $I$ (equivalence can be tested by complete reduction).

### A. Finite Gröbner Bases

If the Basis Algorithm terminates, the resulting set of relations is a Gröbner Basis—for, by the nature of the algorithm, condition (2) is satisfied. Any match for which S-Pol($M$)

does not have a finite $d$-representation will produce a new basis element. The algorithm terminates when this no longer happens.

We have seen that the MBA terminates for (RESOL) and (EB). Thus we may assert that the relations we obtained in those cases form a Gröbner Basis.

It is of interest to note that, since rings of polyomials in commuting variables are Noetherian rings, the Basis Algorithm always terminates in the classical case. Our application to Operator Theory has produced examples in which the Basis Algorithm does not terminate—but which do yield Gröbner Bases.

### B. An Infinite Gröbner Basis

Many of the defining relations in this paper do not give finite Gröbner Bases. However, they may give infinite ones.

We shall illustrate this by proving Theorem V.2.

**Proof of Theorem V.2.** The relations arising from $\mathcal{R}_1^{\mathcal{F}}$ equal those produced by

$$x < x^{-1} < (1-x)^{-1} < S < T \tag{5.1x}$$

union those arising from

$$y < y^{-1} < (1-y)^{-1} < S < T\ . \tag{5.1y}$$

We shall analyze only one of them (5.1x), and since the other has a similar structure a similar argument will prevail. The core of the proof is

**Proposition VI.1.** *The Mora algorithm applied to elements* (5.1x) *with starting relations the defining relations for* $(\ )^{-1}$ *as well as the relation* $Sx - xT$*, yield*

$$\begin{aligned}
\mathrm{Res}_0 &:= x^{-1}x - 1\\
\mathrm{Res}_1 &:= xx^{-1} - 1\\
\mathrm{Res}_2 &:= (1-x)^{-1}x - (1-x)^{-1} + 1\\
\mathrm{Res}_4 &:= (1-x)^{-1}x^{-1} - (1-x)^{-1} - x^{-1}\\
\mathrm{rel}[\mathcal{V}_1] &:= Sx - xT\\
\mathrm{rel}[\mathcal{V}_2] &:= Tx^{-1} - x^{-1}S
\end{aligned} \tag{5.2}$$

together with the three general relations

$$(5.3)\qquad \begin{aligned} A[n] &:= xT^n(1-x)^{-1} - S^n(1-x)^{-1} + S^n \\ C[n] &:= x^{-1}S^n(1-x)^{-1} - T^n(1-x)^{-1} - x^{-1}S^n \\ E[n] &:= (1-x)^{-1}T^n(1-x)^{-1} - (1-x)^{-1}S^n(1-x)^{-1} \\ &\qquad -T^n(1-x)^{-1} + (1-x)^{-1}S^n \ . \end{aligned}$$

These form a Gröbner basis for the ideal of relations.

The general relations GA0 and GA1 are obtained from $\mathrm{rel}[\mathcal{V}_1]$ and $\mathrm{rel}[\mathcal{V}_2]$. The general relations GA2–GA4 are obtained from A, C and E. For convenience we make $A, B$,and $C$ into functions

$$\begin{aligned} A(h) :=& xT(h)(1-x)^{-1} - S(h)(1-x)^{-1} + S(h) \\ C(h) :=& x^{-1}S(h)(1-x)^{-1} - T(h)(1-x)^{-1} - x^{-1}S(h) \\ E(h) :=& (1-x)^{-1}T(h)(1-x)^{-1} - (1-x)^{-1}S(h)(1-x)^{-1} \\ & - T(h)(1-x)^{-1} + (1-x)^{-1}S(h) \end{aligned}$$

We have $A[n](h) = A(h^n)$, etc.

For convenience, we also introduce

$$\begin{aligned} TC[n] &:= T^n x^{-1} - x^{-1}S^n \\ SA[n] &:= S^n x - xT^n \end{aligned}$$

Notice that $TC[n]$ reduces to

$$\sum_{i+j=n-1} T^i \mathrm{rel}[\mathcal{V}_2] S^j$$

and $SA[n]$ reduces to

$$\sum_{i+j=n-1} S^i \mathrm{rel}[\mathcal{V}_1] T^j \ .$$

**Proof of Proposition VI.1.** The (MBA) has been described in terms of “matches”. We denote by M a 6-tuple

$$(f_1, f_2, l_1, r_1, l_2, r_2)$$

where $(l_1, r_1, l_2, r_2)$ is a match for $(\mathrm{lm}(f_1), \mathrm{lm}(f_2))$. Mora [Mora] (Theorem 3.3) shows that a set of polynomials is a Gröbner Basis for the ideal they generate if

$$S\text{-}Pol(M) := l_1 f_1 r_1 - l_2 f_2 r_2$$

has a finite $d$-representation for every $M$ formed from pairs in the set. We will show that the set of relations described above satisfies this criterion.

The only matches among the general relations involve $(E[n], E[m])$, $(E[n], C[m])$, and $(E[n], A[m])$. We could view these in the present notation using powers or describe these matches as

$$(E(h), E(k)),\ (E(h), C(k)), \quad \text{and} \quad (E(h), A(k))\ .$$

Either would do and in this proof we shall stick with powers. We now list each match, $M$, and a finite $d$-representation (see definition sec. VI) for $S\text{-}Pol(M)$.

$$\begin{aligned} M \quad &= (E[n], E[m], (1-x)^{-1}T^m, 1, 1, T^n(1-x)^{-1}) \\ S\text{-}Pol(M) \quad &= -E[m]S^n(1-x)^{-1} + (1-x)^{-1}S^mE[n] \\ &\quad +T^mE[n] - E[n+m] + E[m]S^n \end{aligned}$$

$$\begin{aligned} M \quad &= (E[n], C[m], x^{-1}S^m, 1, 1, T^n(1-x)^{-1}) \\ S\text{-}Pol(M) \quad &= -C[m]S^n(1-x)^{-1} + T^mE[n] \\ &\quad -C[n+m] + C[m]S^n \end{aligned}$$

$$\begin{aligned} M \quad &= (E[n], A[m], xT^m, 1, 1, T^n(1-x)^{-1}) \\ S\text{-}Pol(M) \quad &= -A[m](1-x)^{-1} + S^mE[n] \\ &\quad -A[n+m] + A[m]S^n \end{aligned}$$

The six special relations (5.2) interact with each other to give, by the nature of the Basis Algorithm, relations in the final set. Thus we need only describe interactions between the special relations (5.2) and the general ones (5.3).

$$M = (C[n], \mathrm{Res}_0, x, 1, 1, S^n(1-x)^{-1})$$
$$S\text{-}Pol(M) = -A[n] - \mathrm{Res}_0 S^n$$

$$M = (C[n], \mathrm{Res}_3, (1-x)^{-1}, 1, 1, S^n(1-x)^{-1})$$
$$S\text{-}Pol(M) = -E[n] - \mathrm{Res}_3 S^n + C[n]$$

$$M = (C[n], \mathrm{rel}[\mathcal{V}_2], T, 1, 1, S^n(1-x)^{-1})$$
$$S\text{-}Pol(M) = C[n+1] - \mathrm{rel}[\mathcal{V}_2]S^n$$

$$M = (A[n], \mathrm{Res}_1, x^{-1}, 1, 1, T^n)$$

$$S\text{-}Pol(M) = -C[n]$$

$$M = (A[n], \mathrm{Res}_2, (1-x)^{-1}, 1, 1, T^n(1-x)^{-1})$$
$$S\text{-}Pol(M) = E[n]$$

$$M = (A[n], \mathrm{rel}[\mathcal{V}_1], S, 1, 1, T^n)$$
$$S\text{-}Pol(M) = A[n+1]$$

$$M = (\mathrm{Res}_3, A[n], xT^n, 1, 1, x^{-1})$$
$$S\text{-}Pol(M) = S^n\mathrm{Res}_3 - A[n] - xTC[n] - \mathrm{Res}_0 S^n$$

$$M = (\mathrm{Res}_3, C[n], x^{-1}S^n, 1, 1, x^{-1})$$
$$S\text{-}Pol(M) = T^n\mathrm{Res}_3 - C[n] - TC[n]$$

$$M = (\mathrm{Res}_3, E[n], (1-x)^{-1}T(c), 1, 1, x^{-1})$$
$$S\text{-}Pol(M) = (1-x)^{-1}S^n\mathrm{Res}_3 + T^n\mathrm{Res}_3 - E[n] - (1-x)^{-1}TC[n] - \mathrm{Res}_3 S^n + TC[n]$$

$$M = (\mathrm{Res}_2, A[n], xT^n, 1, 1, x)$$
$$S\text{-}Pol(M) = S^n\mathrm{Res}_2 - A[n]$$

$$M = (\mathrm{Res}_2, C[n], x^{-1}S^n, 1, 1, x^{-1})$$
$$S\text{-}Pol(M) = T^n\mathrm{Res}_2 - C[n] + x^{-1}SA[n] + \mathrm{Res}_1 T^n$$

$$M = (\mathrm{Res}_2, E[n], (1-x)^{-1}T^n, 1, 1, x^{-1})$$
$$S\text{-}Pol(M) = (1-x)^{-1}S^n\mathrm{Res}_2 + T^n\mathrm{Res}_2 - E[n] - (1-x)^{-1}\mathrm{SA}[n] - \mathrm{Res}_2 T^n$$

No other pairs give matches, so condition (2) of the theorem is satisfied.

The authors strongly believe that the Grobner Basis Property holds for the other sets of relations which occur in this paper. We now list evidence for this. In the (NF) case, condition (2) of the Theorem has been verified by machine for all pairs containing basis elements arising from the special rules and all instances of the general rules up to n=12. A general proof like that given above could be given but the greater number of combinations of rules in the (preNF) and (NF) case would make it cumbersome to write.

## VII. SUMMARY OF PRACTICAL RULES YOU MIGHT USE

Here we list in the most condensed form rules that we have generated. Some of these might be worth teaching a functional analysis class. Note that we have replaced resolvents such as $(\lambda - a)^{-1}$ with $(1-x)^{-1}$. This is a normalization easily implemented by dividing $a$ by $\lambda$.

**A. Rules corresponding to the classical resolvent identity**:

(RES) $\qquad x \qquad x^{-1} \qquad (1-x)^{-1}$

$$xx^{-1} \longrightarrow 1 \qquad x^{-1}x \longrightarrow 1$$

$$(1-x)^{-1}x \longrightarrow (1-x)^{-1} - 1$$

$$x(1-x)^{-1} \longrightarrow (1-x)^{-1} - 1$$

$$x^{-1}(1-x)^{-1} \longrightarrow (1-x)^{-1} - x^{-1}$$

$$(1-x)^{-1}x^{-1} \longrightarrow (1-x)^{-1} - x^{-1}$$

**B. Rules for general analytic functions $h$ of $xy$**

$$\text{(GENR)} \qquad = \begin{cases} x \quad x^{-1} \quad (1-x)^{-1} \quad \text{(RES)} \\ h(xy) \quad h(yx) \end{cases}$$

here $h$ is analytic on $\sigma(XY)$, $\sigma(YX)$ yields

$$h(xy)x \longrightarrow xh(yx)$$

$$h(yx)\,x^{-1} \longrightarrow x^{-1}h(xy)$$

$$xh(yx)(1-x)^{-1} \longrightarrow -h(xy) + h(xy)(1-x)^{-1}$$

$$x^{-1}h(xy)(1-x)^{-1} \longrightarrow x^{-1}h(xy) + h(yx)(1-x)^{-1}$$

$$(1-x)^{-1}h(yx)(1-x)^{-1} - (1-x)^{-1}h(xy)(1-x)^{-1}$$

$$\longrightarrow (1-x)^{-1}h(xy) - h(yx)(1-x)^{-1}$$

Note these rules hold for any operators $a$, $b$ replacing $x$ $y$. The rules in A and B have broad application; enough so that teaching them in a general functional analysis class might be an interesting experiment.

**C. Rules special to** $h(s) = (1-s)^{-n}$

These apply to polynomials in

$$x \qquad x^{-1} \qquad (1-x)^{-1} \qquad y \qquad y^{-1} \qquad (1-y)^{-1} \qquad (1-xy)^{-1} \qquad (1-yx)^{-1}$$

which is the same as RES + GENR with $h$ specialized to $K(s) = (1-s)^{-n}$ and yields in addition to the rules in A and C.

$$(1-xy)^{-1}xy \longrightarrow (1-xy)^{-1} - 1$$

$$xy(1-xy)^{-1} \longrightarrow (1-xy)^{-1} - 1$$

$$(1-xy)^{-1}y^{-1} \longrightarrow (1-yx)^{-1}x + x^{-1}$$

$$(1-x)^{-1}y(1-xy)^{-1} \longrightarrow (1-x)^{-1}(1-xy)^{-1} + (1-xy)^{-1} - (1-x)^{-1}$$

together with an infinite class of relations which holds for functions $K(s) = (1-s)^{-n}$ for $n \geq 0$:

$$xK(yx)(1-y)^{-1} \longrightarrow$$
$$K(xy)(1-y)^{-1} + xK(yx) - (1-xy)K(xy)(1-y)^{-1}$$

Note that the following rules depend on alphabetical order.

$$(1-x)^{-1}K(xy)(1-y)^{-1} - (1-x)^{-1}K(xy)(1-y)^{-1}$$
$$\longrightarrow (1-x)^{-1}(1-xy)K(xy)(1-y)^{-1} + K(yx)\{(1-y)^{-1} - 1\} - (1-x)^{-1}K(yx)$$

$$(1-y)^{-1}K(yx)(1-x)^{-1} - (1-y)^{-1}K(xy)(1-x)^{-1} \longrightarrow$$
$$+ (1-y)^{-1}\Sigma(n-1)(1-x)^{-1} + \Sigma(n)(1-x)^{-1} + (1-y)^{-1}\Sigma(n) - \Sigma(n)$$

where $\Sigma(n) = (1-xy)^{-n} + (1-xy)^{-n+1} + \cdots + 1 \quad \text{AND} \quad n \geq 1.$

**D. Rules special to** $K(xy) = (1-xy)^{-n}(1-xy)^{1/2}$

Add $(1-xy)^{1/2}$ and $(1-xy)^{1/2}$ to the list of generators in C.

This yields the same rules as $C$ with

$$K(s) = (1-s)^{-n}(1-s)^{1/2} \qquad \hat{\Sigma}(n) = \Sigma(n)(1-xy)^{1/2}$$

In other words adjoining the square root does nothing fundementally new.

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