A new type of HJBI inequality governing situations where certainty equivalence fails

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Abstract. This paper deals with the generalization of H^∞ control methods to non-linear systems. The classical certainty equivalence principle says that the measurement feedback can be obtained by applying the state feedback law to a certain "worst case" estimate of the state. However, all the results to date relied heavily on the assumption that the "worst case" state is unique. In this paper we study in detail the special case when there are precisely two "worst case" estimates of the state. Our analysis gives not only sufficient but also necessary conditions for the special case of two "worst case" estimates of the state. The key element of our construction is a new "two-point" Hamilton-Jacobi-Isaac-Belman equation which deals with two states at the same time. The results announced here are part of a longer paper which includes more details on smoothness assumption and how to avoid them.

Key Words. Nonlinear control systems, Game theory, Dynamic programming, State estimation

1. The H^∞ Problem Statement

We study the following non-linear system

$$\begin{aligned} \dot{x} &= A(x) + B_1(x)w + B_2(x)u \\ z &= C_1(x) + u \\ y &= C_2(x) + w \end{aligned} \tag{1}$$

where w is disturbance, z is the output, y is the input of the controller and u is the controller output which gets fed into the system. Our objective is:

Find a causal (measurement feedback) control law $y \longrightarrow u$ which makes the system dissipative over a fixed finite horizon:

$$\int_0^T |z(t)|^2 - |w(t)|^2 \mathrm{d}t \le \Phi(x(0)) - \Phi(x(T)) \quad (CTRL)$$

and which makes the system internally stable. Here $\Phi(x(0))$ denotes the energy stored in the unknown initial state $x(0)$ and $\Phi(x(T))$ denotes the energy of the final state $x(T)$.

For linear systems this originated with(Zames and Francis, 1983) and for nonlinear problems with (Ball, Foias, Helton, and Tannenbaum,1987).

We will also need the reversed arrow system:

$$\begin{aligned} \dot{\xi} = &\quad A(\xi) + B_1(\xi)(y - C_2(\xi)) + B_2(\xi)u \\ z = &\quad C_1(x) + u \\ w = &\quad y - C_2(x) \end{aligned} \tag{2}$$

2. The HJBI equation

2.1. The traditional HJBI equation

The HJBI equation equation which governs H^∞ control with measurement feedback for stable systems is

$$\frac{\partial V}{\partial t} + \inf_u \sup_y (\nabla_x V(x) \cdot (A(x) + B_1(x)(y - C_2(x))$$

$$+B_2(x)u) + |C_1(x) + u|^2 - |y - C_2(x)|^2) \tag{3}$$

The solution to (3) and a boundary condition $V(x,T) = \Phi(x)$ is just a value function for the state feedback problem.

In other words,

$$V(x,t) = \int_t^T |C_1(\xi) + u^*(\tau)|^2$$

$$-|y^*(\tau) - C_2(\xi)|^2 d\tau + \Phi(\xi(T)) \tag{4}$$

where $\xi(\tau)$ is the solution to the reverse arrow problem (2) with the initial condition $\xi(t) = x$. The functions u^* and y^* are the extremal values in

$$\inf_u \sup_y \Big(\int_t^T |C_1(\xi) + u^*(\tau)|^2$$

$$-|y^*(\tau) - C_2(\xi)|^2 d\tau + \Phi(\xi(T)) \Big) \tag{5}$$

Even for measurement feedback for unstable systems this equation plays a major role, provided a hypothesis called certainty equivalence (CE) holds. This is because there is a "separation principle" which says that the H^∞ problem breaks in two parts of which (3) is one. This will be discussed later.

We are interested in the case when certainty equivalence fails and what we find is that the key player is a generalization of (3).

2.2. **A new type of HJBI equation**

Given two functions $u(\tau)$ and $y(\tau)$ on interval $\tau \in [t, T]$, consider the game theory problem defined by the cost function

$$COST(u, y, t) = \max\{$$

$$r_1 + \int_t^T |C_1(\xi_1) + u(\tau)|^2$$

$$-|C_2(\xi_1) - y(\tau)|^2 d\tau + \Phi(\xi_1(T)),$$

$$r_2 + \int_t^T |C_1(\xi_2) + u(\tau)|^2$$

$$-|C_2(\xi_2) - y(\tau)|^2 d\tau + \Phi(\xi_2(T))\} \qquad (6)$$

which y tries to maximize and u tries to minimize. Here r_1, r_2 are fixed real numbers. Here $\xi_1(\tau)$ and $\xi_2(\tau)$ are the solutions to the state trajectory equations with fixed initial conditions $\xi_1(t) = x_1$ and $\xi_2(t) = x_2$. Thus one is given x_1, x_2, r_1, r_2; then the COST depends on input functions u, y.

Suppose there are policies $u^*(x_1, x_2, r_1, r_2, \tau)$ and $y^*(x_1, x_2, r_1, r_2, \tau)$ at which the inf and sup are achieved at:

$$\mathcal{V} = \inf_u \sup_y COST(u, y, t) \qquad (7)$$

Now define

$$V_1(x_1, x_2, r_1, r_2, t) = \int_t^T |C_1(\xi_1) + u^*(\tau)|^2$$

$$-|y^*(\tau) - C_2(\xi_1)|^2 d\tau + \Phi(\xi_1(T)) \qquad (8)$$

where $u^*(\tau) = u^*(x_1, x_2, r_1, r_2, \tau)$ and $y^*(\tau) = y^*(x_1, x_2, r_1, r_2, \tau)$ are the extreme values of u and y in (7). Similarily, define

$$V_2(x_1, x_2, r_1, r_2, t) = \int_t^T |C_1(\xi_2) + u^*(\tau)|^2$$

$$-|y^*(\tau) - C_2(\xi_2)|^2 d\tau + \Phi(\xi_2(T)) \qquad (9)$$

Then the following obvious relation holds:

$$V_2(x_1, x_2, r_1, r_2, t) = V_1(x_2, x_1, r_2, r_1, t)$$

There are two basically different types of situations.

$$r_1 + V_1(x_2, x_1, r_2, r_1, t) = r_2 + V_2(x_2, x_1, r_2, r_1, t) (BP)$$

and

$$\mathcal{V} = r_1 + V_1 \quad \text{or} \quad \mathcal{V} = r_2 + V_2 \qquad (TRD)$$

the balanced performance case and the traditional case. The second situation is degenerate in that $\mathcal{V}$ takes on the form $r_1 + V_1$ or the form $r_2 + V_2$. This puts us in the traditional case of section 2.1. While it is possible and even likely that a single control problem will produce a mixture of these cases we focus on the pure version of case (BP). One might think that since (BP) calls for equality of two functions it is rare, however bear in mind that r_1+V_1, r_2+V_2 are produced by optimizing the max of these functions over u, y, a process which tends to equalize their values.

We found for the (BP) case that the role of the traditional HJBI equation is subsumed by the " two-point" HJBI equation based on a Pre-Hamiltonian defined by:

$$\mathcal{H}(V, x_1, x_2, r_1, r_2, t, u, y) =$$
$$= |C_1(x_1) + u|^2 - |y - C_2(x_1)|^2$$
$$+\frac{\partial V_1}{\partial x_1} \cdot [A(x_1) + B_1(x_1)(y - C_2(x_1)) + B_2(x_1)u] +$$
$$\frac{\partial V_1}{\partial x_2} \cdot [A(x_2) + B_1(x_2)(y - C_2(x_2)) + B_2(x_2)u] +$$
$$+\frac{\partial V_1}{\partial r_1}(|C_1(x_1) + u|^2 - |y - C_2(x_1)|^2) +$$
$$\frac{\partial V_1}{\partial r_2}(|C_1(x_2) + u|^2 - |y - C_2(x_2)|^2)$$

for a function V of (x_1, x_2, r_1, r_2, t).

Theorem 2.1 Suppose the functions V_1, V_2 are defined by (8) and (9) and suppose they are smooth.

Then V_1 and V_2 satisfy the equation:

$$\inf_u \sup_y \max\{\frac{\partial V_1}{\partial t} + \mathcal{H}(V_1, x_1, x_2, r_1, r_2, t, u, y),$$

$$\frac{\partial V_2}{\partial t} + \mathcal{H}(V_2, x_1, x_2, r_1, r_2, t, u, y)\} = 0 \qquad (10)$$

at points x_1, x_2, r_1, r_2, t where the (BP) condition holds.

They also satisfy the boundary condition

$$\begin{aligned} V_1(x_1, x_2, r_1, r_2, T) &= \Phi(x_1) \\ V_2(x_1, x_2, r_1, r_2, T) &= \Phi(x_2) \end{aligned} \qquad (11)$$

This theorem will be proved in section 6.

3. The role of "two-point" Hamiltonian in H^∞ control

We briefly review the theory of non-linear H^∞ control. For the sake of simplicity and generality we shall follow the article (James, 1995) in this volume . See it for much more detail and for references.

3.1. *Information state*

We begin with the notion of information state. For fixed policies $u(\tau)$ and $y(\tau)$ the information state is defined as

$$p(x,t) = \Phi(\xi(0)) +$$

$$\int_0^t |C_1(\xi) + u(\tau)|^2 - |y(\tau) - C_2(\xi)|^2 d\tau \quad (12)$$

where $\xi(\tau)$ is the solution to the inverse arrow system with $\xi(t) = x$. We will be treating p as a known function of unknown state x. The definition (12) implies that the dynamics for the information state is given by

$$\frac{\partial p(x,t)}{\partial t} = Q(-\nabla_x p, u, y, x) \quad (13)$$

where

$$Q(L,u,y,x) = L\cdot(A(x) + B_2(x)u + B_1(x)(y - C_2(x)))$$
$$+ |C_1(x) + u|^2 - |y - C_2(x)|^2$$

Our Q corresponds to F in (James, 1995). Following James we introduce the function

$$W(p,t) = \inf_u \sup_y$$

$$\sup_x \{p(x,T) + \Phi(x(T)) : p(x,t) = p(x)\} \quad (14)$$

which is in fact the value function of the game which arises from (CTRL).

The dynamic programming principle implies then that W satisfies

$$\inf_{u \in R^N} \sup_{y \in R^N} \frac{dW(p(\cdot,t),t)}{dt} = 0 \quad (15)$$

for solutions $p(\cdot,t)$ to (13). If W is smooth we can apply the chain rule to conclude that

$$\frac{dW(p,t)}{dt} = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial p}\frac{dp}{dt} \quad (16)$$

In the view of (13), equation (15) can be written as:

$$\frac{\partial W(p,t)}{\partial t} +$$

$$\inf_u \sup_y DirD[W, p, Q(-\nabla_x p, u, y, x)] = 0 \quad (17)$$

which since W is often not smooth holds in a viscosity sense. Here $DirD[W,p,h]$ stands for the one sided directional derivative of W, in variable p, in direction h. One sided implies that $DirD[W,p,h]$ may not equal $DirD[W,p,-h]$.

Roughly speaking (James, 1995) says that properties of W and p determine if a solution to (CTRL) exists. Also it gives high level formulas for controllers coming from W and p. Thus the key issue becomes computation of W and p. This paper addresses computation of W and says that in broad circumstances it can be done from (3) or (10). See section 5 for more details.

3.2. *Certainty equivalence for one point estimator*

It is known that W can be computed using the solution V to (3). under what is called the certainty equivalence assumption. See (Bernhard, Basar, 1991; Krener, 1993; James, Baras, 1994; Chichka, Speyer, 1994; Van der Schaft, 1994). There is work on relaxing the CE assumption by (c.f. James, 1995) and (Dydinsky, Basar, Bernhard, 1993) and by Chichka and Speyer.

In the case of measurement feedback, the state x is not known; so we try to find a "worst" x. That is an x which maximizes the sum of the past input-output balance, $p(x)$, and the future optimal input-output balance, which is:

$$Int(x,u,y,t) =$$

$$\int_t^T |C_1(\xi) + u(\tau)|^2 - |y(\tau) - C_2(\xi)|^2 d\tau + \Phi(\xi(T))$$

where $\xi(t) = x$. In other words we consider

$$\max_x \inf_u \sup_y (p(x) + Int(x,u,y,t)) \quad (18)$$

The classical certainty equivalence principle is based on the following assumption:

Assumption 1 *For every t there is only one maximizing x in (18)*

If we consider the state feedback problem as a game , then it has the value function $V(x,t)$ satisfying HJBI equation (3). Moreover a simple argument (see [JB]) shows that (18) is equal to the cost function W, i.e

$$W(p,t) = \max_x (p(x) + V(x,t)) \quad (19)$$

and therefore the optimal measurement feedback is given by $u^*_{\text{state}}(\hat{x})$ where $\hat{x}$ is the maximizing x in (19) and u^*_{state} is the optimal u in (3).

3.3. *Certainty equivalence for multi-point estimators*

In this paper we study the case of non-unique maximizing x in (18). We make the following

Assumption 2 *For every t there are precisely two maximizing x's in (18)*

In this case (18) can be written

$$\max_{x\in\{\hat{x}_1,\hat{x}_2\}} \inf_u \sup_y (p(x) + Int(x,u,y,t)) \tag{20}$$

Therefore the optimal u and y depend on both $\hat{x}_1$ and $\hat{x}_2$ as well as $p(\hat{x}_1)$ and $p(\hat{x}_2)$ and (20) can be written as:

$$\max\{p(\hat{x}_1) + Int(\hat{x}_1,u,y,t),$$

$$p(\hat{x}_2) + Int(\hat{x}_2,u,y,t)\} \tag{21}$$

In other words, (18) is equal to

$$\max_{\hat{x}_1,\hat{x}_2} \max\{p(\hat{x}_1) + V_1(\hat{x}_1,\hat{x}_2,p(\hat{x}_1),p(\hat{x}_2)),$$

$$p(\hat{x}_2) + V_2(\hat{x}_1,\hat{x}_2,p(\hat{x}_1),p(\hat{x}_2))\} \tag{22}$$

with V_1, V_2 defined in section 2.2.

This assumption by itself is fairly strong. In practice one would expect to see the cases where for some periods of time we have unique maximizing x and for other periods of time two maximizing x's. Then the two approaches, the classical with unique estimate and ours, would have to be combined.

4. The Relationship between W and $V's$

In this section we prove that under assumption 2 the cost function W is equal to (22) provide V_1, V_2 satisfying (10).

Theorem 4.1 Suppose smooth functions V_1 and V_2 satisfy equation (10) at points where (BP) holds. Suppose they also satisfy condition (11). Suppose that for given information state $p(x,t)$ satisfying (13) and for a given time t_0 there is precisely one pair $(\hat{x}_1,\hat{x}_2)$ at which the max of the two functions

$$\max_{x_1,x_2} \max\{p(x_1,t_0)+V_1(x_1,x_2,p(x_1,t_0),p(x_2,t_0)),$$

$$p(x_2,t_0) + V_2(x_1,x_2,p(x_1,t_0),p(x_2,t_0))\}$$

is achieved and that at $(\hat{x}_1,\hat{x}_2)$ the two functions are equal.

Then the function

$$\tilde{W}(p,t) = \max_{x_1,x_2} \max\{p(x_1)+V_1(x_1,x_2,p(x_1),p(x_2)),$$

$$p(x_2) + V_2(x_1,x_2,p(x_1),p(x_2))\} \tag{23}$$

satisfies equation (15) at $p = p(\cdot,t_0)$ and $t = t_0$.

Conversely, if $\tilde{W}(p,t)$ defined as above, satisfies (15) then V_1, V_2 satisfy (10).

Remark 4.2 Note that if a function $M(p,t)$ satisfies equation (15) along a trajectory of information state and satisfies the condition $M(p,T) = \sup_x(p(x)+\Phi(x))$ (which is satisfied by W, see (14)) then $M(p,t) = W(p,t)$. This follows from the fact that both functions are constants along the trajectory and have the same values at end points.

Thus Theorem 4.1 combined with Theorem 2.1 gives us a concrete formula for W under the "two point" assumption.

Proof of Theorem 4.1 The assumptions of the theorem imply that at $(\hat{x}_1,\hat{x}_2)$ derivatives with respect to x_1 and x_2 of $p(x_1,t_0) + V_1(x_1,x_2,p(x_1,t_0),p(x_2,t_0),t_0)$ are equal to zero:

$$\nabla_x p(\hat{x}_1) + \frac{\partial V_1}{\partial x_1} + \frac{\partial V_1}{\partial r_1}\nabla_x p(\hat{x}_1) = 0 \tag{24}$$

$$\frac{\partial V_1}{\partial x_2} + \frac{\partial V_1}{\partial r_2}\nabla_x p(\hat{x}_2) = 0 \tag{25}$$

The same is true for V_2, i.e.

$$\nabla p(\hat{x}_2) + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_2}{\partial r_2}\nabla_x p(\hat{x}_2) = 0 \tag{26}$$

$$\frac{\partial V_2}{\partial x_1} + \frac{\partial V_2}{\partial r_1}\nabla_x p(\hat{x}_1) = 0 \tag{27}$$

The total directional derivative of $\tilde{W}(p,t)$ in variables (p,t) in the direction $(h,1)$ is given by:

$$DirD[\tilde{W},(p,t),(h,1)] = \max\{$$

$$\frac{\partial V_1}{\partial t} + h(\hat{x}_1) + \frac{\partial V_1}{\partial r_1}h(\hat{x}_1) + \frac{\partial V_1}{\partial r_2}h(\hat{x}_2),$$

$$\frac{\partial V_2}{\partial t} + h(\hat{x}_2) + \frac{\partial V_2}{\partial r_2}h(\hat{x}_2) + \frac{\partial V_2}{\partial r_1}h(\hat{x}_1)\} \tag{28}$$

with the partial derivatives of V_1,V_2 evaluated at $\hat{x}_1, \hat{x}_2, p(\hat{x}_1, t_0), p(\hat{x}_2, t_0), t_0$ We use this to evaluate

$$\inf_u \sup_y \frac{d\tilde{W}(p,t_0)}{dt} =$$

$$DirD[\tilde{W},(p,t),(Q(-\nabla_x p,u,y,x),1)] =$$

$$\inf_u \sup_y \max\{\frac{\partial V_1}{\partial t} + (1+\frac{\partial V_1}{\partial r_1})Q(-\nabla_x p,u,y,\hat{x}_1)+$$

$$\frac{\partial V_1}{\partial r_2}Q(-\nabla_x p,u,y,\hat{x}_2),$$

$$\frac{\partial V_2}{\partial t} + (1+\frac{\partial V_2}{\partial r_2})Q(-\nabla_x p,u,y,\hat{x}_2)+$$

$$\frac{\partial V_2}{\partial r_1}Q(-\nabla_x p,u,y,\hat{x}_1)\} \tag{29}$$

In order to abbreviate the expressions above we shall introduce the notation

$$b(x) = |y - C_1(x)|^2 - |C_2(x) + u|^2 \tag{30}$$

(i.e. $b(x)$ denotes the difference between energy out and energy in) and

$$f(x) = A(x) + B_1(x)w + B_2(x)u \tag{31}$$

i.e. $f(x)$ is the dynamics of the state x. Equation (24) implies that

$$(1+\frac{\partial V_1}{\partial r_1})Q(-\nabla_x p,u,y,\hat{x}_1) =$$

$$(1+\frac{\partial V_1}{\partial r_1})Q((1+\frac{\partial V_1}{\partial r_1})^{-1}\nabla_{x_1}V_1,u,y,\hat{x}_1) =$$

$$= \nabla_{x_1}V_1 \cdot f(\hat{x}_1) + b(x_1) + \frac{\partial V_1}{\partial r_1}b(x_1)$$

Equation (25) implies that

$$\frac{\partial V_1}{\partial r_2}Q(-\nabla_x p,u,y,\hat{x}_2) =$$

$$-\frac{\partial V_1}{\partial r_2}\nabla_x p(\hat{x}_2)\cdot f(\hat{x}_2) + \frac{\partial V_1}{\partial r_2}b(x_2) =$$

$$= \frac{\partial V_1}{\partial x_2}\cdot f(\hat{x}_2) + \frac{\partial V_1}{\partial r_2}b(x_2)$$

Therefore (29) becomes

$$\inf_u \sup_y \max\{\frac{\partial V_1}{\partial t} + \nabla_{x_1}V_1\cdot f(\hat{x}_1) + b(x_1)+$$

$$\frac{\partial V_1}{\partial r_1}b(x_1) + \frac{\partial V_1}{\partial x_2}\cdot f(\hat{x}_2) + \frac{\partial V_1}{\partial r_2}b(x_2),$$

$$\frac{\partial V_2}{\partial t} + \nabla_{x_2}V_2\cdot f(\hat{x}_2) + b(x_2)+$$

$$\frac{\partial V_2}{\partial r_2}b(x_2) + \frac{\partial V_2}{\partial x_1}\cdot f(\hat{x}_1) + \frac{\partial V_2}{\partial r_1}b(x_1)\} \tag{32}$$

which is exactly the left side of equation (10) which is equal to zero. Thus we have just shown that the total derivative of W is zero and therefore (15) holds. ♠

5. Equations for the optimal controller

The main point of Theorem 4.1 is that the value function W, originaly defined as $\inf_u \sup_y \max_x$ of a certain quantity, is actually equal to $\max_{x_1,x_2} \inf_u \sup_y$. Comparing the definition of W, definitions of V_1, V_2 and formula (23) implies immediately that the optimal u in the case of Assumption 2 is given by

$$u^*(\hat{x}_1, \hat{x}_2, p(\hat{x}_1,t), p(\hat{x}_2,t), t) \tag{33}$$

where $\{\hat{x}_1, \hat{x}_2\}$ is the maximizing pair in (23) and u^* is the optimal value of u in the two-point HJBI equation (10).

Let us write $u^*(p)$ instead of (33) in the sequel. The dynamics of the optimal controller is given by equation

$$\frac{\partial p}{\partial t} = Q(-\nabla_x p, u^*(p), y, x)$$

where p play the role of the state variable of the controller. Here y is the controller input, $u^*(p)$ is the controller output and p is the state variable of controller.

6. Proof of Theorem 2.1

For every $\tau \in [t,T]$ consider the quantity

$$\max\{r_1 + p(\xi_1,\tau) + V_1(\xi_1,\xi_2,p(\xi_1,\tau),p(\xi_2,\tau)),$$

$$r_2 + p(\xi_2,\tau) + V_2(\xi_1,\xi_2,p(\xi_1,\tau),p(\xi_2,\tau))\} \tag{34}$$

where $\xi_1(\tau), \xi_2(\tau)$ are solutions to (2) with initial conditions $\xi_1(t) = x_1, \xi_2(t) = x_2$ and with $u = u^*$ and $y = y^*$ giving inf and sup in (6). The information state p in this case is given by

$$p(\xi,\tau) = \int_t^\tau |C_1(\zeta(s)) + u^*(s)|^2 - |y^*(s) - C_2(\zeta(s))|^2 ds$$

where $\zeta(s)$, is the solution to (2) for $t \le s \le \tau$ which satisfies the initial condition $\zeta(\tau) = \xi(\tau)$. Note that (34) is independent of τ, because it is equal to the cost $COST(u^*, y^*, t)$ (see (6)) of optimal policies u^*, y^*.

Now we are going to compute the total derivative of (34) with respect to τ at $\tau = t$. This derivative is, of course, equal to zero.

Since we only deal with x_1, x_2, r_1, r_2, t where (BP) holds, the two functions inside of max in (34) are equal at $\tau = t$ and therefore we can interchange $\frac{d}{d\tau}$ and max:

$$\max\{\frac{d}{d\tau}\Big|_{\tau=t}(r_1 + p(\xi_1,\tau)+$$

$$V_1(\xi_1,\xi_2,p(\xi_1,\tau),p(\xi_2,\tau)),\tau)),$$
$$\frac{d}{d\tau}\Big|_{\tau=t}(r_2+p(\xi_2,\tau)+$$

$$V_2(\xi_1,\xi_2,p(\xi_1,\tau),p(\xi_2,\tau)),\tau))\}=0 \qquad (35)$$

The total derivative of the first function can be written as (using notation from (30) and (31)) :

$$\frac{d}{d\tau}\Big|_{\tau=t}(r_1+p(\xi_1,\tau)+$$
$$V_1(\xi_1,\xi_2,p(\xi_1,\tau),p(\xi_2,\tau)),\tau))=$$
$$b(\xi_1)+\frac{\partial V_1}{\partial t}+\frac{\partial V_1}{\partial x_1}f(\xi_1)+$$

$$\frac{\partial V_1}{\partial x_2}f(\xi_2)+\frac{\partial V_1}{\partial r_1}b(\xi_1)+\frac{\partial V_1}{\partial r_2}b(\xi_2) \qquad (36)$$

Evaluating at $\tau = t$ gives us $\xi_1 = x_1$ and $\xi_2 = x_2$ and therefore (36) can be written as:

$$b(x_1)+\frac{\partial V_1}{\partial t}+\frac{\partial V_1}{\partial x_1}f(x_1)+\frac{\partial V_1}{\partial x_2}f(x_2)+$$

$$\frac{\partial V_1}{\partial r_1}b(x_1)+\frac{\partial V_1}{\partial r_2}b(x_2) \qquad (37)$$

This together with the analogous computation for the second function in (35) gives us exactly equation (10). ♠.

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