NONCOMMUTATIVE CONVEXITY ARISES FROM LINEAR MATRIX INEQUALITIES.

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November 2005

Abstract. This paper concerns polynomials in g noncommutative variables $x = (x_1, \ldots, x_g)$, inverses of such polynomials, and more generally noncommutative "rational expressions" with real coefficients which are formally symmetric and "analytic near 0". The focus is on rational expressions $r = r(x)$ which are "matrix convex" on the unit ball; i.e., those rational expressions r such that if $X = (X_1, \ldots, X_g)$ is a g-tuple of $n \times n$ symmetric matrices satisfying

$$I_n - \left(X_1^2 + \cdots + X_g^2\right) \text{ is positive definite}$$

and Y is also, then the symmetric matrix

$$t\, r(X) + (1-t)\, r(Y) - r(tX + (1-t)Y) \text{ is positive semidefinite}$$

for all numbers t, $0 \leq t \leq 1$. This article gives a complete classification of matrix convex rational expressions (see Theorem 3.3) by representing such r in terms of a symmetric "linear pencil"

$$\mathcal{L}_\gamma(x) := I_d - \sum_j \mathcal{A}_j x_j + \begin{pmatrix} 0_{d-1} & 0 \\ 0 & -1 + \gamma - \ r(0) \end{pmatrix}$$

in the noncommuting variables x_j, where $\mathcal{A}_j$ are symmetric $d \times d$ matrices. Namely, for γ a real number, $\gamma - r$ is a Schur complement of the linear pencil $\mathcal{L}_\gamma$. Moreover, given a matrix convex r, the set consisting of g tuples X of $n \times n$ symmetric matrices

$$\{X : r(X) - \gamma I \text{ is negative definite}\} \tag{0.1}$$

has component containing 0 which is the same as the "negativity set",

$$\{X : \mathcal{L}_\gamma(X) \text{ is negative definite}\} \tag{0.2}$$

for $\mathcal{L}_\gamma$. Conditions like $L_\gamma(X)$ is negative definite are known as linear matrix inequalities (LMIs) in the engineering literature and arguably the main advance in linear systems theory in the 1990's was the introduction of LMI techniques. In this language what we have shown in (0.1) vs. (0.2) is that the set of solutions to a "convex matrix inequality" with noncommutative unknowns is the same as the set of solutions to some LMI.

In many engineering systems problems convexity would have all of the advantages of LMIs. Indeed convexity guarantees that solutions are global and

Partly supported by the NSF and the Ford Motor Co.
Partly supported by the NSF grant DMS-0140112.
Partially supported by the Israel Science Foundation (Grant No. 322/00).

convexity bodes well for reliability of the numerics. Since LMIs have a structure which is seemingly much more rigid than convexity, there is the continual hope that a convexity based theory will be more far reaching than LMIs. But will it? There are two natural situations: one where the unknowns are scalars and one where the unknowns are matrices appearing in formulas which respect matrix multiplication. These latter problems mathematically yield expressions with noncommutative unknowns and they arise in engineering systems problems which are "dimensionless" in the sense that they scale "automatically with dimension" (as do most of the classics of control theory.) That is the case we study here and the result stated above suggests the surprising conclusion that for dimensionless systems problems convexity offers no greater generality than LMIs. Indeed the result proves this for a class of model problems. Furthermore, we show that existing algorithms together with algorithms described here *construct* the LMIs above which are equivalent to the matrix inequalities based on the given matrix convex rational function r.

In a very different direction we prove that a symmetric polynomial p in g noncommutative symmetric variables has a symmetric determinantal representation, namely, there are symmetric matrices $A_0, \dots, A_g$ in $\mathbb{SR}^{d\times d}$ with A_0 invertible such that

$$\det p(X) \;=\; \det\left(A_0 - L_A(X)\right) \tag{0.3}$$

for each X a g-tuple of symmetric $n \times n$ matrices. Of course taking $n = 1$ implies immediately that a (commuting variables) polynomial p on $\mathbb{R}^g$ has a symmetric determinantal representation. For $g = 2$ much stronger commutative results can be obtained using tools of algebraic geometry but these do not seem to generalize to the higher dimensional case; on the other hand, a *nonsymmetric commutative determinantal representation* for any g is due to Valiant ("universality of determinant" in algebraic complexity theory).

Our determinantal representation theorem is a bi-product of the theory of systems realizations of noncommutative rational functions and can be read independently of much of the rest of the paper.

While the notion of noncommutative rational functions is standard, the equivalence relation we use on rational expressions in our construction, based on evaluating rational expressions on matrices, is new and gives a new approach to noncommutative rational functions.

1. Readers Guide

We have written the paper so that it may be read on several levels.

Section 2 gives an informal introduction to rational functions. Our main results are stated in Section 3. Thus readers interested in only the statements of the results on the structure of convex NC rational functions and convex matrix inequalities need read just the next two sections. They may also wish to read Section 15 which provides some motivation from the point of view of linear systems theory. Many of the ideas of the proofs can be gotten by reading only through Section 7, while the full proofs are considerably heavier and require reading most of this paper.

The reader interested only in determinantal representations may go to Section 14 after first scanning Subsections 2.4.3, 4.1, and 4.2, Subsubsection 2.1.1, and Lemma 13.1.

At the end of Section 3 we suggest several possibilities for reading (parts of) the remainder of the paper.

2. An Introduction to NC Rational Functions

At first glance this notation section may look formidable to many readers. We offer the reassurance that much of it lays out formal properties of noncommutative rational functions which merely capture manipulations with functions on matrices which are very familiar to systems engineers, matrix theorists and operator theorists. People in these areas are advised, on first reading, to move quickly to §3, which describes our main results.

2.1. NC Linear Pencils.

Throughout this paper $x = (x_1, \dots, x_g)$ denotes g noncommutative indeterminates. Given a matrix W with entries W_{ij} and a variable x_ℓ, let $Wx_\ell = x_\ell W$ denote the matrix with entries given by

$$(Wx_\ell)_{ij} = W_{ij}x_\ell.$$

Given $m \times d$ matrices $M_1, \dots, M_g$, define L_M by

$$L_M(x) := M_1x_1 + \cdots + M_gx_g.$$

A $m \times d$ **NC linear pencil** (in g indeterminates) is an expression of the form

$$M(x) := M_0 + L_M(x)$$

where $L_M(x) = M_1x_1 + \dots M_gx_g$ and $M_0, M_1, \dots, M_g$ are $m \times d$ matrices. As an example, for

$$M_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad M_1 := \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \quad M_2 := \begin{pmatrix} 5 & 4 \\ 4 & 2 \end{pmatrix},$$

the pencil is

$$M(x) = \begin{pmatrix} 1 + 3x_1 + 5x_2 & 2x_1 + 4x_2 \\ 2x_1 + 4x_2 & -1 + x_1 + 2x_2 \end{pmatrix}.$$

Sometimes we refer to the pencil as $M_0, M_1, \dots, M_g$. Frequently we multiply all of the matrices M_j by a single matrix G or W, namely,

$$GL_M(x)W = L_{GMW}(x)$$

where $GMW := (GM_1W, \dots, GM_gW)$.

Note the common term linear pencil is a misnomer in that *linear pencils are actually affine linear*, that is, the pencil $M(x)$ is linear if and only if $M_0 = 0 = M(0)$.

We usually deal with **symmetric NC linear pencils**, enough so that whenever we write a pencil with coefficients denoted by A, namely L_A, we are referring to A_j which are $d \times d$ (real) symmetric matrices.

2.1.1. *Pinned Pencils.* We say that a $m \times d$ linear pencil M is **pinned** if the matrices M_j^{T}, for $j = 1, \dots, g$, have a common nonzero null vector $\eta \in R^m$. We emphasize that $M_0\eta^T$ is not required to be 0. We call η a **vector pinning the pencil**. In the sequel when pinning is an issue, it will turn out that the common null space of the M_j has dimension at most one, so that we shall call η **the** vector pinning the pencil. In the favorable circumstance that the pencil is not pinned, we say it is **unpinned**.

2.1.2. *Evaluation of Linear Pencils.* Denote the $n \times n$ matrices with real entries by $\mathbb{R}^{n\times n}$, and the subspace of symmetric $n \times n$ matrices by $\mathbb{SR}^{n\times n}$. Similarly, denote g-tuples of (resp. symmetric) $n \times n$ matrices by $(\mathbb{R}^{n\times n})^g$ (resp. $(\mathbb{SR}^{n\times n})^g$) and let $\mathbb{S}^g$ be the disjoint union

$$\mathbb{S}^g = \cup_{n=0}^{\infty} (\mathbb{SR}^{n\times n})^g.$$

Given $m \times d$ matrices $M_0, M_1, \dots, M_g$ and a g-tuple $X = (X_1, \dots, X_g) \in (\mathbb{R}^{n\times n})^g$, let

$$L_M(X) = M_1 \otimes X_1 + \cdots + M_g \otimes X_g$$

and

$$M(X) = M_0 \otimes I_n + L_M(X).$$

The tensor product in the expressions above is the usual tensor product of matrices. Thus we have reserved the tensor product notation for the tensor product of matrices and have eschewed the strong temptation of using $M \otimes x_\ell$ in place of Mx_ℓ when x_ℓ is one of the noncommuting indeterminates.

2.2. **NC Rational Functions.** NC rational functions are described in detail in Appendix 16. That process has a certain unavoidable heft to it, and we hope to make this paper accessible to people in areas (such as systems engineering) where NC rational functions are manipulated successfully without too much formalism. Thus we give here a brief version of our formalism which turns on evaluation of rational expressions at tuples of symmetric matrices. The fact that evaluation based equivalence and properties such as Lemma 2.2 are new forces much of the discussion.

2.2.1. *A Few Words about Words.* $\mathcal{W}_g$ denotes the free semi-group on the g symbols $\{\chi_1, \dots, \chi_g\}$. As always, we let $x_1 \dots, x_g$ be g noncommuting formal variables, and for a word $w = \chi_{i_1} \dots \chi_{i_k} \in \mathcal{W}_g$ we define $x^w = x_{i_1} \dots x_{i_k}$.

Occasionally we consider variables which are formal transposes x_j^{T} of a variable x_j, and words in all of these variables $x_1, \dots, x_g, x_1^{\mathrm{T}}, \dots, x_g^{\mathrm{T}}$, often called the words in x, x^{T}. If w is in $\mathcal{W}_g$, then w^{T} denotes the transpose of a word w. For example, given the word (in the x_j's) $x^w = x_{j_1} x_{j_2} \dots x_{j_n}$, the involution applied to x^w is $(x^w)^{\mathrm{T}} = x_{j_n}^{\mathrm{T}} \dots x_{j_2}^{\mathrm{T}} x_{j_1}^{\mathrm{T}}$, which, if the variables x_k are symmetric, is $x^{(w^{\mathrm{T}})} = x_{j_n} \dots x_{j_2} x_{j_1}$. In this paper, unless said otherwise, the variables x_k satisfy $x_k^{\mathrm{T}} = x_k$ for $k = 1, \dots, g$, i.e., they are symmetric.

2.2.2. *The Ring of NC Polynomials.* $\mathbb{R}\langle x_1, \dots, x_g\rangle :=$ the ring of noncommutative polynomials over $\mathbb{R}$ in the noncommuting variables $x_1, \dots, x_g$. We often abbreviate $\mathbb{R}\langle x_1, \dots, x_g\rangle$ by $\mathbb{R}\langle x\rangle$. When the variables x_k are symmetric the algebra $\mathbb{R}\langle x\rangle$ maps to itself under the involution ${}^{\mathrm{T}}$. Occasionally we work with non-symmetric variables, and the algebra of polynomials in them is denoted

$$\mathbb{R}\langle x_1, \dots, x_g, x_1^{\mathrm{T}}, \dots, x_g^{\mathrm{T}}\rangle \qquad \text{or} \qquad \mathbb{R}\langle x, x^{\mathrm{T}}\rangle.$$

2.2.3. *Polynomial Evaluations.* If p is an NC polynomial in the symmetric variables $x_1, \dots, x_g$ and $X = (X_1, X_2, \dots, X_g)$ is in $(\mathbb{SR}^{n\times n})^g$, the evaluation $p(X)$ is defined by simply replacing x_j by X_j. Note that, for $Z_n = (0_n, 0_n, \dots, 0_n) \in (\mathbb{SR}^{n\times n})^g$ where each 0_n is the $n \times n$ zero matrix, $p(0_n) = I_n \otimes p(0_1)$. In particular, $p(0_n)$ is invertible for all n or no n. Because of this simple relationship, in the sequel we will often simply write $p(0)$ with the size n unspecified.

2.2.4. *Rational Functions and Rational Expressions.* We shall define the notion of a NC rational function analytic at 0 in terms of rational expressions.

We use recursion to define the notion of a **NC rational expression** r **analytic at** 0 and its value $r(0)$ at 0. This class includes polynomials and $p(0)$ is the value of p at 0 as in the previous subsubsection. If $p(0)$ is invertible, then p is invertible, this inverse is a NC rational expression analytic at 0, and $p^{-1}(0) = p(0)^{-1}$. Formal sum and products of NC rational expressions analytic at 0 with the value at 0 are defined accordingly. Finally, a NC rational expression r analytic at 0 can be inverted provided $r(0) \neq 0$, this inverse is an NC rational expression, and $r^{-1}(0) = r(0)^{-1}$.

A difficulty is that two different expressions, such as

$$r_1 = x_1(1 - x_2 x_1)^{-1} \quad \text{and} \quad r_2 = (1 - x_1 x_2)^{-1} x_1 \tag{2.1}$$

can be converted to each other with such operations. Thus one needs to specify an equivalence relation on rational expressions. The one we use here is classical and uses formal power series expansions

$$\sum_{w\in\mathcal{W}_g} r_w x^w$$

of NC rational expressions around 0. As an example, consider the operation of inverting a polynomial. If p is a NC polynomial and $p(0) \neq 0$, write $p = p(0) - q$

where $q(0) = 0$, then the inverse p^{-1} is the series expansion $r = \frac{1}{p(0)} \sum_k (\frac{q}{p(0)})^k$. Clearly, taking successive products, sums and inverses allows us to obtain a NC formal power series expansion for any NC rational expression analytic at 0.

We say that two NC rational expressions r_1 and r_2 analytic at 0 are **power series equivalent** if their series expansion around 0 are the same. For example, the series expansion for the functions r_1 and r_2 above are

$$\sum_{k=0} x_1(x_2x_1)^k \quad \text{and} \quad \sum_{k=0} (x_1x_2)^k x_1. \tag{2.2}$$

These are the same series, so r_1 and r_2 are power series equivalent.

A **noncommutative rational function analytic at** 0 is an equivalence class $\mathfrak{r}$ under the power series equivalence relation and the **series expansion for** $\mathfrak{r}$ is the series expansion of any representative. The set of these equivalence classes is denoted $\mathbb{R}\langle x \rangle_{Rat0}$. We often shorten the phrase noncommutative rational function analytic at 0 to **NC rational function** or simply **rational function**, since it is the class of functions we treat in this paper, and we typically use German (Fraktur) font to denote NC rational functions.

2.3. **Evaluation and Domains of NC Functions.** In semi-algebraic geometry one considers regions in $\mathbb{R}^g$ where a given rational function in g commuting variables takes positive or nonnegative values. We shall be interested in an analogous type of noncommutative semi-algebraic geometry. Here we evaluate NC rational functions on g-tuples of symmetric matrices and consider tuples which make the rational functions take values which are positive semidefinite. We shall work with the natural order on matrices generated by the cone of positive semidefinite matrices, namely, for $A, B \in \mathbb{SR}^{n\times n}$

$$A \prec B \quad \text{means} \quad B - A \text{ is positive definite, and}$$

$$A \preceq B \quad \text{means} \quad B - A \text{ is positive semidefinite.}$$

2.3.1. *The Formal Domain of a Rational Expression.* The formal domain in $(\mathbb{SR}^{n\times n})^g$ of a NC rational expression r, denoted $\mathcal{F}(n)_{r,\text{for}}$, is defined inductively. If p is a polynomial, then it is is all of $(\mathbb{SR}^{n\times n})^g$. If r is the inverse of the polynomial p, then the formal domain of r is is $\{X \in (\mathbb{SR}^{n\times n})^g : p(X) \text{ is an invertible matrix}\}$. The formal domain of a general NC rational expression r, is equal to the intersection of formal domains $\mathcal{F}(n)_{r_j,\text{for}}$ for the rational expressions r_j and appropriate domains of inverses of the r_k which appear in the expression r. Let

$$\mathcal{F}_{r,\text{for}} = \cup_{n\geq 1} \mathcal{F}(n)_{r,\text{for}}.$$

The following proposition collects some observations about $\mathcal{F}_{r,\text{for}}$.

Proposition 2.1. *Let r be a rational expression analytic at* 0.

(Rep) *For each n, the domain $\mathcal{F}(n)_{r,for}$ is closed with respect to unitary conjugation: If $X = (X_1, \ldots, X_g) \in \mathcal{F}(n)_{r,for}$ and U is an $n \times n$ unitary matrix, then $UXU^{\mathrm{T}} = (UX_1U^{\mathrm{T}}, \ldots, UX_gU^{\mathrm{T}}) \in \mathcal{F}(n)_{r,for}$.*

(Sum) *The domain $\mathcal{F}_{r,for}$ is closed with respect to direct sums: If $X \in \mathcal{F}(n)_{r,for}$ and $Y \in \mathcal{F}(m)_{r,for}$, then $X \oplus Y = (X_1 \oplus Y_1, \dots, X_g \oplus Y_g) \in \mathcal{F}(n+m)_{r,for}$. Here*

$$X_j \oplus Y_j = \begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix}.$$

(Zopen) *$\mathcal{F}(n)_{r,for}$ is a non-empty Zariski open subset of $(\mathbb{SR}^{n\times n})^g$ containing 0.*

(0open) *There exists an $\varepsilon > 0$ such that if $X \in (\mathbb{SR}^{n\times n})^g$ with*

$$X_1^2 + \cdots + X_g^2 \prec \varepsilon I,$$

then X is in the formal domain of r. Here ε is independent of n.

Proof. The first three claims are obvious. The proof of the (0open) property is in Lemma 16.5.

2.3.2. *The Domain and Evaluation of a Rational Function.* It is clear how to evaluate a NC rational expression r on any $X \in \mathcal{F}(n)_{r,\text{for}}$. We can use this to define an equivalence on noncommutative rational expressions which we call evaluation equivalence. Two NC rational expressions r and $\tilde{r}$ analytic at 0 are **evaluation equivalent** provided $r(X) = \tilde{r}(X)$ for each n and each X in the Zariski open set $\mathcal{F}(n)_{r,\text{for}} \cap \mathcal{F}(n)_{\tilde{r},\text{for}}$.

The following lemma shows that evaluation equivalence is the same as power series equivalence we have defined in §2.2.4. After proving the lemma, we shall simply refer to it in the sequel as **equivalence.** Notice that evaluation equivalence can be also defined for noncommutative rational expressions which are not necessarily analytic at the origin, leading to an explicit construction of the whole skew field of noncommutative rational functions, see §16.6.

Lemma 2.2. *The noncommutative rational expressions $\tilde{r}$ and r analytic at 0 are power series equivalent if and only if they are evaluation equivalent.*

Remark 2.3. The fact that both r and $\tilde{r}$ are analytic at 0 means that for each dimension n, the 0 matrix g-tuple is in the intersection of their domains. Without the requirement that r and $\tilde{r}$ are analytic at 0 it is possible that for certain n one or both of the domains $\mathcal{F}(n)_{r,\text{for}}$ or $\mathcal{F}(n)_{\tilde{r},\text{for}}$ could be empty.

Proof. First suppose rational expressions r and $\tilde{r}$ are power series equivalent, so they have the same power series expansion, and this series expansion converges in some neighborhood $\mathcal{N}$ of 0 in $(\mathbb{SR}^{n\times n})^g$. It may be assumed that $\mathcal{N} \subset \mathcal{F}(n)_{r,\text{for}} \cap \mathcal{F}(n)_{\tilde{r},\text{for}}$. It follows that $(r - \tilde{r})(X) = 0$ on $\mathcal{N}$. Since $r - \tilde{r}$ has a power series expansion in $gn(n-1)/2$ real variables which is convergent[1] near 0, it vanishes identically on $\mathcal{F}(n)_{r,\text{for}} \cap \mathcal{F}(n)_{\tilde{r},\text{for}}$. This holds for every n. Hence r and $\tilde{r}$ are evaluation equivalent.

The other implication is the content of Proposition 16.7 item 5 proved in §16. □

[1]the classical name for these is a **real analytic function**

Given an NC rational function $\mathfrak{r}$, define **the (algebraic) domains** $\mathcal{F}(n)_{\mathfrak{r}}$ **and** $\mathcal{F}_{\mathfrak{r}}$ **of** $\mathfrak{r}$ to be

$$\mathcal{F}(n)_{\mathfrak{r}} = \bigcup_{n\geq 1}\{\mathcal{F}(n)_{r,\mathrm{for}} : r \text{ is a rational expression for } \mathfrak{r}\}$$

and

$$\mathcal{F}_{\mathfrak{r}} = \cup_{n\geq 1}\mathcal{F}(n)_{\mathfrak{r}}$$

respectively. In view of Lemma 2.2, given $X \in \mathcal{F}_{\mathfrak{r}}$ the **evaluation** $\mathfrak{r}(X)$ is unambiguously defined by choosing a rational expression r for $\mathfrak{r}$ for which $X \in \mathcal{F}_{r,\mathrm{for}}$ and declaring $\mathfrak{r}(X) = r(X)$.

The connected component of $\mathcal{F}(n)_{\mathfrak{r}}$ containing 0 is denoted $\mathcal{F}(n)^0_{\mathfrak{r}}$ and $\mathcal{F}^0_{\mathfrak{r}}$ is

$$\mathcal{F}^0_{\mathfrak{r}} := \cup_{n\geq 1}\ \mathcal{F}(n)^0_{r}$$

which we call **the 0 component of the domain of** $\mathfrak{r}$.

Remark 2.4. We emphasize that when we write $\mathfrak{r} = 0$ meaning, the rational function $\mathfrak{r}$ is 0, then any rational expression r representing $\mathfrak{r}$ has the property that for $X \in \mathcal{F}_{r,\mathrm{for}}$, we have $r(X) = 0$. This follows immediately from Lemma 2.2.

2.3.3. *An Alternate Domain.* The definition of the domain of the rational function $\mathfrak{r}$, while natural, is a bit clumsy. For instance, while $\mathcal{F}(n)_{\mathfrak{r}}$ is both an open subset of $(\mathbb{SR}^{n\times n})^g$ and invariant under unitary conjugation, we do not know if $\mathcal{F}_{\mathfrak{r}}$ is closed with respect to direct sums of matrices. This (see §6.1 for the definition) is a key property required in our proofs.

The following notion of domain deserves mention, and while not essential, it is a convenience in the proof of Lemma 2.7 below. Lemma 2.2 says that all rational expressions r for the rational function $\mathfrak{r}$ determine the same rational function on $(\mathbb{SR}^{n\times n})^g$, namely $\mathfrak{r}$. Accordingly, it is sensible to define **the analytic domain** $\mathcal{AF}(n)_{\mathfrak{r}}$ **of** $\mathfrak{r}$ to be the domain of real analyticity of this rational function and we also define

$$\mathcal{AF}_{\mathfrak{r}} := \cup_{n\geq 1}\mathcal{AF}(n)_{\mathfrak{r}},$$

which we call **the analytic domain of** $\mathfrak{r}$. Notice that for any NC rational expressions r representing $\mathfrak{r}$ we have $\mathcal{F}(n)_{r,\mathrm{for}} \subseteq \mathcal{AF}(n)_{\mathfrak{r}}$. Thus, *the (algebraic) domain of* $\mathfrak{r}$ *is contained in the analytic domain of* $\mathfrak{r}$. We do not know if these two domains are the same.

2.3.4. *Symmetric Rational Functions.* A rational function $\mathfrak{r}$ is **symmetric** if its values $\mathfrak{r}(X)$ are symmetric; i.e., provided $\mathfrak{r}(X)^{\mathrm{T}} = \mathfrak{r}(X)$ for each $X \in \mathcal{F}_{\mathfrak{r}}$. Writing the power series expansion for $\mathfrak{r}$ as $\sum \mathfrak{r}_w x^w$, note that $\mathfrak{r}$ is symmetric if and only if $\mathfrak{r}_w = \mathfrak{r}_{w^{\mathrm{T}}}$; this follows from Proposition 16.7 item 5 in Appendix 16. (Alternatively, we can define a rational function $\mathfrak{r}$ to be symmetric if it coincides with its transpose, $\mathfrak{r} = \mathfrak{r}^{\mathrm{T}}$, see Section 16.3.)

2.4. **Matrix Valued NC Rational Expressions and Functions.** The notion of rational expression is broadened by using matrix constructions. Indeed, this more general notion is often used in this paper.

2.4.1. *Matrix-valued Rational Expressions.* Matrix-valued NC rational expressions analytic at 0 are defined by analogy to (scalar-valued) rational expressions. A **matrix-valued NC polynomial** is a NC polynomial with matrix coefficients. All matrix-valued NC polynomials are matrix-valued rational expressions. If P is a square matrix-valued NC polynomial and $P(0)$ is invertible, then P has an inverse P^{-1} whose formal domain is

$$\mathcal{F}_{P^{-1},\mathrm{for}} = \{X \in \mathbb{S}^g : P(X) \text{ is invertible}\}.$$

Matrix-valued NC rational expressions R_1 and R_2 can be added and multiplied whenever their dimensions allow, with the formal domain of the sum and product equal to the intersection of the formal domains. Finally, a square matrix-valued NC rational expression R has an inverse as long as $R(0)$ is invertible. (See §16.4 for details.)

A $m_1 \times m_2$ matrix-valued NC rational expression analytic at 0 has a power series expansions whose coefficients are $m_1 \times m_2$ matrices. Matrix-valued NC rational expressions R_1 and R_2 are equivalent provided they have the same power series expansion and a **matrix-valued NC rational function analytic at** 0 is an equivalence class of matrix-valued NC rational expressions. In particular, the definition of **rational expression analytic at** 0 is now **amended** to mean 1×1 matrix-valued rational expressions analytic at 0. Notice that the evaluation of a matrix-valued NC rational expression or power series on a g-tuple of matrices uses tensor substitution of matrices as explained for pencils in Section 2.1.2.

We shall use the phrase **scalar rational expression analytic at** 0 if we want to emphasize the absence of matrix constructions. Often when the context makes the usage clear we drop adjectives such as scalar, 1×1, matrix rational, matrix of rational and the like. Indeed, it is shown in Section 16.4 (see Proposition16.9 and Theorem 16.10) that a $m_1 \times m_2$-matrix valued noncommutative rational function is in fact the same as a $m_1 \times m_2$ **matrix of noncommutative rational functions**, and furthermore, any matrix valued noncommutative rational function can be a represented by a matrix of scalar rational expressions "near" any point in its domain.

2.4.2. *Symmetric Matrix NC Rational Expressions and Functions.* A square matrix R of scalar NC rational expressions or a square matrix-valued NC rational expression is called **symmetric** if $R(X) = R(X)^{\mathrm{T}}$ for each X in $\mathcal{F}(n)_{R,\mathrm{for}}$. The notion of symmetric for a rational function $\mathfrak{R}$ is, of course, defined similarly. Note that such an $\mathfrak{R}$ has a symmetric formal Taylor series expansion:

$$\mathfrak{R}(x) = \sum_{w \in \mathcal{W}_g} \mathfrak{R}_w x^w = \sum_{w \in \mathcal{W}_g} x^w \mathfrak{R}_w$$

with $\mathfrak{R}_w^{\mathrm{T}} = \mathfrak{R}_{w^{\mathrm{T}}}$; as in the scalar case, this follows from Proposition 16.7 item 5 in Appendix 16.

2.4.3. *Descriptor Realizations, More on Evaluation of NC Linear Pencils, and an Example.* To illustrate matrix NC rational functions and other definitions, consider the (1×1) rational expression given by

$$(x) = D + C(J - L_A(x))^{-1}B. \tag{2.3}$$

Here $A_1, \dots, A_g$ are $d \times d$ matrices, $B \in \mathbb{R}^d$, $C \in \mathbb{R}^{1\times d}$, and J is a signature matrix, meaning $J = J^{\mathrm{T}}$ and $J^{\mathrm{T}}J = I$. We call expressions of the form (2.3) **descriptor realizations**. A descriptor realization is called *symmetric* if $A_1, \dots, A_g$ are symmetric matrices and $C = B^{\mathrm{T}}$ and $D = D^T$. Symmetric descriptor realizations play a major role in this paper.

Example 2.5. Here is an example of a 1×1 rational expression in two variables obtained as a descriptor realization.

$$\begin{aligned}(x) &= \begin{pmatrix}1 & 0\end{pmatrix}\left(I - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}x_1 - \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}x_2\right)^{-1}\begin{pmatrix}1\\ 0\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 - x_1 & -x_2\\ -x_2 & 1 - x_1\end{pmatrix}^{-1}\begin{pmatrix}1\\ 0\end{pmatrix}.\end{aligned}$$

An NC symmetric rational expression representing the same NC rational function as is

$$r = (1 - x_1)^{-1} + (1 - x_1)^{-1}x_2\left((1 - x_1) - x_2(1 - x_1)^{-1}x_2\right)x_2(1 - x_1)^{-1}.$$

□

The tensor product notation (already used in $L_A(X)$) provides a convenient way of expressing the evaluation

$$(X) = D \otimes I_n + (C \otimes I_n)[J \otimes I_n - L_A(X)]^{-1}(B \otimes I_n). \tag{2.4}$$

at $X \in (\mathbb{SR}^{n\times n})^g$. Here I denotes the $n \times n$ identity where n is chosen to match the size of X. We often abbreviate $B \otimes I_n$ to B and $C \otimes I_n$ to C, although this is an abuse of notation.

Computing the formal power series expansion, and thus the equivalence class (rational function) to which the descriptor realization belongs, is straightforward.

$$\begin{aligned}(x) = B^{\mathrm{T}}(I - JL_A(x))^{-1}JB \sim &\sum_{n\geq 0} B^{\mathrm{T}}(JL_A(x))^n JB\\ =&B^{\mathrm{T}}JB + \sum_{j=1}^{g} B^{\mathrm{T}}JA_jJBx_j + \dots.\end{aligned}$$

This uses $A_jBx_j = A_jx_jB$.

Example 2.6. We return to the rational expression in Example 2.5.

Note it is straightforward to compute the power series expansion. Also the formal domain of the rational expression is, by definition exactly those $X = (X_1, X_2) \in (\mathbb{SR}^{n\times n})^2$ for which

$$\begin{pmatrix}I - X_1 & -X_2\\ -X_2 & I - X_1\end{pmatrix}$$

is invertible, and for such X

$$(X) = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} I - X_1 & -X_2 \\ -X_2 & I - X_1 \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

□

We will need the following property of pencils.

Lemma 2.7. *Suppose $M_0, M_1, \dots, M_g$ are $d \times d$ matrices and let $\mathfrak{R}$ denote the rational function determined by the rational expression $R(x) := (M_0 - L_M(x))^{-1}$. If M_0 is invertible, then*

$$\mathcal{AF}_{\mathfrak{R}} = \mathcal{F}_{\mathfrak{R}} = \mathcal{F}_{R,for} = \{X \in \mathbb{S}^g \colon (M_0 - L_M(X)) \text{ is invertible}\}.$$

(Here and in the proof we abuse the notation by writing M_0 instead of $M_0 \otimes I_n$ where n is the size of the matrices X.)

Proof. Evidently

$$\mathcal{F}_{R,\text{for}} := \{X \colon (M_0 - L_M(X)) \text{ is invertible}\} \subseteq \mathcal{F}_{\mathfrak{R}} \subseteq \mathcal{AF}_{\mathfrak{R}}.$$

Thus it remains to verify the reverse inclusion. Let X^0 in $(\mathbb{SR}^{n\times n})^g$ satisfy $M_0 - L_M(X^0)$ is not invertible, $\det(M_0 - L_M(X^0)) = 0$. We pursue a contradiction by assuming that X^0 is in $\mathcal{AF}_{\mathfrak{R}}$, that is, that the matrix-valued rational function $R(X)$ can be analytically continued to a neighborhood of X^0. Then the scalar rational function $\det R(X)$ can be also analytically continued to a neighborhood of X^0. Since $(M_0 - L_M(X))R(X) = I$, we have $\det(M_0 - L_M(X)) \det R(X) = 1$ for $X \in \mathcal{F}^n_{R,\text{for}}$ and we can continue this equation analytically to $X = X^0$ getting the contradiction $0 = 1$. □

The formal domain of a descriptor realization in (2.3) is by definition the set $\{X : J - L_A(X) \text{ is invertible}\}$. Also, in keeping with an earlier remark, Theorem 16.10 in Appendix 16 says that this is the domain of a symmetric descriptor realization regarded as a *scalar* NC rational function.

2.5. **Growth.** We say $\mathfrak{r}$ **has at most order** k **growth at infinity** if for each tuple X in $(\mathbb{SR}^{n\times n})^g$,

$$\lim_{t\to\infty} \frac{\mathfrak{r}(tX)}{t^{k+1}} = 0.$$

$\mathfrak{r}$ **is said to have at most linear growth at infinity** provided $k = 1$. To apply this definition we need that for each X there is a $T_X \in \mathbb{R}$ such that for $t > T_X$, $tX \in \mathcal{F}_{\mathfrak{r}}$. This holds for any rational function $\mathfrak{r}$ analytic at zero since $\mathfrak{r}$ admits a (monic) descriptor realization (see Section 4 below), hence the domain of $\mathfrak{r}$ contains $\{X \colon \det(I - L_A(X)) \neq 0\}$; therefore for each X, $tX \in \mathcal{F}_{\mathfrak{r}}$ except for finitely many values of t.[2]

[2] Here is a direct way to see this, without using realization theory. Let r be a rational expression representing $\mathfrak{r}$, then the complement $\mathfrak{S}^n$ to $\mathcal{F}_{r,for}$ in $(\mathbb{SR}^{n\times n})^g$ is a Zariski closed subset not containing the origin. In particular, for each X the line $\{tX \colon t \in \mathbb{R}\}$ is not contained in $\mathfrak{S}^n$ and therefore intersects $\mathfrak{S}^n$ in finitely many points.

3. Main Theorems

A symmetric (possibly matrix valued) NC rational expression r is called **matrix convex** on a domain $\mathcal{D} \subset \mathcal{F}_{r,\mathrm{for}}$ provided for each fixed n whenever X, Y *in* $\mathcal{F}(n)_{r,\mathrm{for}}$ both are on a line segment which lies entirely inside $\mathcal{D}$, the inequality

$$t\, r(X) + (1-t)\, r(Y) - r(tX + (1-t)Y) \text{ is positive semidefinite} \tag{3.1}$$

holds for all $0 \leq t \leq 1$. For example, r is **matrix convex near** 0, if there is an $\epsilon > 0$ such that r is matrix convex on the ball $X_1^2 + \cdots + X_g^2 \prec \varepsilon I$ of symmetric matrix g tuples X of any dimension.

A symmetric rational function $\mathfrak{r}$ is **matrix convex** on a domain $\mathcal{D} \subset \mathbb{S}^g$, if there is a rational expression r for $\mathfrak{r}$ which is defined and matrix convex on $\mathcal{D}$.

Remark 3.1. An alternate, and slightly less restrictive, definition of matrix convexity for a rational function $\mathfrak{r}$ is obtained by simply replacing the term "rational expression" with "rational function" in (3.1). While we do not use this definition in the paper, we could have done so with little modification of the proofs.

3.1. LMI Theorem.

Our first main theorem, as described in the abstract, asserts that whenever there is convexity there is an associated LMI.

Theorem 3.2. *If $\mathfrak{r}$ is a noncommutative scalar symmetric rational function in symmetric variables x which is matrix convex near 0, then $\mathfrak{r}$ is matrix convex on $\mathcal{F}_{\mathfrak{r}}^0$ and the set $\mathcal{F}_{\mathfrak{r}}^0$ is the positivity set for a monic noncommutative symmetric pencil; i.e., there is a pencil $\mathcal{L}(x) = I - \sum A_j x_j$, with each A_j symmetric and $\mathcal{F}_{\mathfrak{r}}^0 = \mathcal{F}_{\mathcal{L}^{-1}}^0$, that is;*

$$\mathcal{F}_{\mathfrak{r}}^0 = \{X \in \mathbb{S}^g : \mathcal{L}(X) \text{ is positive definite}\}.$$

Moreover, there are symmetric matrices $\mathcal{A}_0, \mathcal{A}_j$ for $j = 1, \dots, g$ and M, such that for each $\gamma > \mathfrak{r}(0)$ the matrix $\mathcal{A}_0 + \gamma M$ is positive semidefinite and (the component of 0 of) the set

$$\{X \in \mathbb{S}^g : \mathfrak{r}(X) - \gamma I \text{ is negative definite}\},$$

which is all solutions to the matrix inequality $\gamma I \succ \mathfrak{r}(X)$, equals all solutions $X \in \mathbb{S}^g$ of the linear matrix inequality

$$\mathcal{L}_\gamma(X) := L_{\mathcal{A}}(X) - (\mathcal{A}_0 + \gamma M) \text{ is negative definite.}$$

Theorem 3.2 follows from the next theorem, Theorem 3.3 below, and is proved in §3.2.3. A formula for $\mathcal{L}_\gamma$ is given by equation (3.3) and it is important to note that, according to Theorem 3.3 and in view of the convexity hypothesis in Theorem 3.2, the J term in equation (3.3) is I. For a discussion of algorithms which compute the ingredients of (3.3) see §3.2.4.

3.2. **Monic Butterfly Realization Theorem.** We say a scalar $\mathfrak{r}$ has a **butterfly realization** provided there is a rational expression for $\mathfrak{r}$ of the form

$$(x) = {}_0 + {}_1(x) + \ell(x)\ell(x)^{\mathrm{T}} + \Lambda(x)(J - L_A(x))^{-1}\Lambda(x)^{\mathrm{T}} \tag{3.2}$$

where

(1) J is a $d \times d$ signature matrix,
(2) A_j are $d \times d$ symmetric matrices,
(3) ${}_0$ is a scalar,
(4) ${}_1(x)$ is 1×1 valued and linear in x,
(5) $\ell(x)$ is $1 \times k$ valued and linear in x,
(6) $\Lambda(x)$ is $1 \times d$ valued and affine linear in x.

That is, ℓ, ${}_1$ and Λ are NC linear pencils of appropriate dimension with ${}_1(0) = 0 = \ell(0)$. In the case that $J = I$ we call the realization **monic**.

Note that the butterfly realization expresses $-\gamma$ as a Schur complement of the linear pencil

$$\mathcal{L}_\gamma(x) := \begin{pmatrix} -1 & 0 & \ell(x)^{\mathrm{T}} \\ 0 & -(J - L_A(x)) & \Lambda(x)^{\mathrm{T}} \\ \ell(x) & \Lambda(x) & {}_0 - \gamma + {}_1(x) \end{pmatrix} \tag{3.3}$$

for any real number γ.

A special case of the butterfly realization is the symmetric descriptor realization; it is the case where ${}_1 = 0 = \ell$ and Λ is a constant (independent of x). This was introduced in §2.4.3. Another special case, which we call a **pure butterfly realization**, has $\Lambda(x)$ linear in x, that is, $\Lambda_0 = 0$.

There is a certain amount of non-uniqueness in such realizations. For example, take $J = I, A = 0$, then

$$(x) = {}_0 + {}_1(x) + \ell(x)\ell(x)^{\mathrm{T}} + \Lambda(x)I_d\Lambda(x)^{\mathrm{T}} \tag{3.4}$$

so the quadratic term of can be expressed using Λ with $\ell = 0$ or using ℓ with $\Lambda = 0$ or mixtures of ℓ and Λ.

Also in Lemma 8.2 and 8.3 we show that if $\mathfrak{r}$ is given by a butterfly realization, then $\mathfrak{r}$ has a pure butterfly realization.

3.2.1. *Minimality.* Write $\Lambda(x) := \Lambda_0 + \sum_{j=1}^{g} \Lambda_j x_j$. For the butterfly realization **observability** means

$$\Lambda_j(JA)^w v = 0 \quad \textit{for all } j = 0, \dots, g$$

for all words w implies $v = 0$. Note $\Lambda_j(JA)^w v = 0$ and $\Lambda_j J(AJ)^w Jv = 0$ are equivalent. **Controllability** means the span of

$$((JA)^w)^{\mathrm{T}}(\Lambda_j)^{\mathrm{T}} \quad \textit{for all } j = 0, \dots, g.$$

is $\mathbb{R}^d$. The realization is called **minimal** if it is both controllable and observable. By symmetry controllability and observability are equivalent.

The realization is called **pinned (resp. unpinned)** provided the pencil $J - L_A(x)$ is pinned (resp. unpinned). There is a certain amount of uniqueness in minimal unpinned butterfly realizations sufficient to prove the $J = I$ portion of Theorem 3.3 below. The precise statement and proof are given in §4.5.

3.2.2. *Representation of NC Convex Rational Functions.* Our main theorem characterizing matrix convex NC rational functions is

Theorem 3.3. *Let* $\mathfrak{r}$ *denote a symmetric noncommutative scalar rational function in symmetric variables* x*.*

Monic Butterfly Realization: *If* $\mathfrak{r}$ *is matrix convex near 0, then* $\mathfrak{r}$ *has a minimal unpinned butterfly realization with* J *equal to the identity. More specifically*

$$(x) = {}_0 + {}_1(x) + \ell(x)\ell(x)^{\mathrm{T}} + \Lambda(x)(I_d - L_A(x))^{-1}\Lambda(x)^{\mathrm{T}}, \tag{3.5}$$

where ℓ *and* ${}_1$ *are linear in* x*, and* $\Lambda(x)$ *is affine linear in* x*, that is,* ${}_1(0) = \ell(0) = 0$*.*

Pure Butterfly Realization: *If* $\mathfrak{r}$ *has a minimal unpinned butterfly realization with* $J = I$*, then* $\mathfrak{r}$ *has a minimal unpinned pure butterfly realization with* $J = I$*.*

Singularities: *If the minimal unpinned butterfly realization in (3.5) is either pure* ($\Lambda_0 = 0$) *or satisfies* $\Lambda(x) := \Lambda_0$*, then* $\mathcal{F}^0_{\mathfrak{r}}$*, the* 0 *component of the domain of the function* $\mathfrak{r}$ *it realizes, equals* $\mathcal{P}$ *defined by*

$$\mathcal{P} := \bigcup_{n \geq 1} \mathcal{P}^n$$

where

$$\mathcal{P}^n := \ \{X \in (\mathbb{SR}^{n\times n})^g : (I_{dn} - L_A(X)) \ is\ positive\ definite\} \tag{3.6}$$

for each n*.*

Convexity Region: *A function* $\mathfrak{r}$ *with a monic butterfly realization (3.5) is matrix convex on* $\mathcal{P}$*. In particular, if* $\mathfrak{r}$ *is matrix convex near* 0*, then the* 0 *component of the domain of* $\mathfrak{r}$ *is convex.*

Growth: *If* $\mathfrak{r}$ *is matrix convex near zero, then:*

1. *Any NC rational* $\mathfrak{r}$ *analytic at 0 has the property: for each* $X \in \mathbb{S}^g$ *there is a* $T_X \in \mathbb{R}$ *such that for* $t > T_X$*,* $\mathfrak{r}(tX)$ *is defined. This was observed in §2.5.*
2. *The growth of* $\mathfrak{r}$ *at infinity is at most second order.*
3. *If* $\mathfrak{r}$ *has at most linear growth at* ∞*, then* $\mathfrak{r}$ *has a realization of the form (3.5) with* $\ell = 0$ *and* $\Lambda(x) = \Lambda_0$ *a constantmatrix.*
4. *If* $\mathfrak{r}$ *has at most order 0 growth at* ∞*, then we may take* $\ell = {}_1 = 0$ *and* $\Lambda(x) = \Lambda_0$ *a constant matrix, that is, (3.5) is a symmetric descriptor realization for* $\mathfrak{r}$*.*
5. *The last term of the butterfly realization (3.5) may be left off if and only if* $\mathfrak{r}$ *is a polynomial of degree* 2 *or less.*

The growth conclusion, item 5 of Theorem 3.3 on polynomials is the main result, Theorem 3.1, of [HM04], indeed we have produced a new proof here. In turn, this specialized to matrix convex polynomials in one variable is due to Ando [A79]. Various forms of matrix convexity in one variable extend back for over 60 years, see [L34] [K36].

The proof of Theorem 3.3 consumes practically all of this paper, so instead of launching into it we turn now to uniqueness issues and to consequences including the proof of Theorem 3.2.

Proposition 3.4. *If $\mathfrak{r}$ is a symmetric noncommutative scalar rational function in symmetric variables x which is matrix convex near 0, then every minimal unpinned pure butterfly realization (3.4) of $\mathfrak{r}$ has $J = I$.*

If $\mathfrak{r}$ is a symmetric noncommutative scalar rational function in symmetric variables x which is matrix convex near 0, then every minimal unpinned descriptor realization (if any exists) of $\mathfrak{r}$ has $J = I$.

Proof. The proof uses Proposition 4.4 which is a variation on the usual uniqueness of minimal transfer function realizations up to similarity. It says, in part, if

$$(x) = {}_0 + {}_1(x) + \ell(x)\ell(x)^{\mathrm{T}} + \Lambda(x)(J - L_A(x))^{-1}\Lambda(x)^T$$

and

$$(x) = \tilde{}_0 + \tilde{}_1(x) + \tilde{\ell}(x)\tilde{\ell}(x)^{\mathrm{T}} + \tilde{\Lambda}(x)(\tilde{J} - L_{\tilde{A}}(x))^{-1}\tilde{\Lambda}(x)^T,$$

are both minimal unpinned pure butterfly realizations, then the state spaces have the same dimension and there is an invertible matrix S so that $S^{\mathrm{T}} J S = \tilde{J}$.

On the other hand, the first part of Theorem 3.3 says that we can choose $J = I$. With this choice, it follows that $\tilde{J} = S^{\mathrm{T}} S$ and therefore, as $\tilde{J}$ is a positive semidefinite signature matrix, we obtain $\tilde{J} = I$.

The descriptor realization conclusion follows by a similar argument, but appeals to Proposition 4.3 which is the descriptor version of Proposition 4.4. A fundamentally easier proof of the descriptor conclusion is that it is the content of Proposition 7.1. □

3.2.3. *Specific LMIs and a Proof of Theorem 3.2.* Theorem 3.2 follows from Theorem 3.3 and the result discussed in §8.5. If $\mathfrak{r}$ is matrix convex near 0, then, by Theorem 3.3, the rational function $\mathfrak{r}$ has a minimal unpinned butterfly realization as in equation (3.5). Moreover,

$$\mathcal{F}_{\mathfrak{r}}^0 = \{X \in \mathbb{S}^g : \ I - L_A(X) \succ 0\}.$$

Here $Y > 0$ means the symmetric matrix Y is positive semidefinite. The result in Subsection 8.5 says that is matrix convex on $\mathcal{F}_{\mathfrak{r}}^0$ and this means that $\mathfrak{r}$ is matrix convex there as well.

Suppose we are given an NC symmetric rational function $\mathfrak{r}$ which is matrix convex even in a small region near 0. We pick a parameter $\gamma \in \mathbb{R}$, $\gamma \geq 0$. Suppose we want to find (if possible) $X \in \mathbb{S}^g$ in the component $\mathcal{G}_\gamma^0$ containing 0 of the set

$$\mathcal{G}_\gamma := \{X : \mathfrak{r}(X) \prec \gamma I\}.$$

The butterfly realization associates this with an LMI problem (as we now formalize in Lemma 3.5). Consequently, a half dozen or so numerical packages are available for numerical solution of the problem, for example, two of the most popular packages are [GNLC95] and [S99]. These are based on semidefinite programming methods which originated in [NN94].

Lemma 3.5. *The pencil $\mathcal{L}_\gamma$ in (3.3) associated by the butterfly realization to $\mathfrak{r}$ satisfies*

$$\mathcal{G}_\gamma^\circ = \{X \in \mathbb{S}^g : \ \mathcal{L}_\gamma(X) \text{ is negative definite}\}.$$

Proof. The upper diagonal 2×2 block of $\mathcal{L}_\gamma$ is negative definite if and only if $J - L_A(X)$ is positive definite. We have $J = I$ and by the singularity conclusion in Theorem 3.3 the matrix $I - L_A(X)$ is positive definite for all X in $\mathcal{F}^0_\mathfrak{r}$, thus it is positive definite on $\mathcal{G}_\gamma^\circ$. Now $\ - \gamma$ is the Schur complement pivoting on the upper diagonal 2×2 block of $\mathcal{L}_\gamma$, and so $\mathcal{L}_\gamma(X)$ is negative definite if and only if $\ (X) - \gamma I$ is negative definite. □

The lemma says that converting the matrix inequality $\{X \in \mathbb{S}^g : \ r(X) \prec \gamma I\}$ to an LMI follows from constructing a butterfly realization. So we discuss algorithms for this construction. One possible approach to producing a descriptor realization for r is to use Hankel operators as in [S61] [F74a] [BGMprept]. While this proves existence of a descriptor realization algorithmic constructions have never been fully worked out. However, close to implementation is the following.

3.2.4. *Algorithm to Produce an LMI.* We present the algorithm in three parts. Combining (1), (2), and (3) below lays out a theoretical framework and algorithms, a significant part of which have been implemented, for going from a matrix convex r to a LMI.

We are given a symmetric NC rational function r:

1. A construction [Sprept], due to N. Slinglend (a graduate student at UCSD), uses something like continued fractions and reminds one of circuit realization constructions. This algorithm produces a minimal descriptor realization, but J might not be I and the realization might be pinned. It has an implementation due to J. Shopple under NCAlgebra, a noncommutative algebra package which runs under Mathematica.

2. *Descriptor-Butterfly Algorithm.* In the course of proving Theorem 3.3 we give an algorithm, Algorithm §8.4.

> *Given a minimal symmetric descriptor realization of a symmetric NC rational function $\mathfrak{r}$ which is matrix convex near 0, Algorithm §8.4 produces a minimal unpinned butterfly realization with $J = I$.*

The algorithm could be implemented in NCAlgebra.

The algorithm works as follows. If the minimal descriptor realization is already unpinned, then $J = I$ by Proposition 3.4. On the other hand, if $J = I$ and the realization is pinned, then the pinning space splits the realization and the desired realization (actually descriptor) is obtained. In the remaining case, it turns out that the pinning space has dimension exactly one and J has exactly one negative eigenvalue. Moreover, in this case there is a rigid structure leading to the construction for obtaining the desired realization, see §7.

3. Also an algorithm given in [CHSY03] and implemented in NCAlgebra determines if a given NC rational function $\mathfrak{r}$ is or is not matrix convex near 0.

We include the caveat that Algorithm (1)–(2)–(3) when implemented would require generalization to be at the level of doing control engineering problems.

3.3. **Readers Guide Redux.** Most of this article is devoted to proving Theorem 3.3.

The proof of Theorem 3.3 is divided into three parts and the reader who reads only the first part can get many of the main ideas. This first part, §4, 5, 6, 7 proves the monic butterfly realization conclusion of Theorem 3.3 for r under a special type of unpinned hypothesis. The second part, Section 8 removes the unpinned hypothesis. The third part §9, 10, 11, 12 prove the singularities conclusion of Theorem 3.3.

The paper also contains a determinantal representation for both commutative and noncommutative polynomials which, as already noted, is largely independent of the rest of the paper. Indeed, the reader that has gotten this far can now go to Section 14 after scanning Subsection 4.2 and Lemma 13.1.

Now we give more detail. We begin, in Section 4, with system realizations for NC multi-variable rational functions, extending the classical work of Schutzenberger [S61]. M. Fliess [F74a] subsequently used Hankel operators effectively in such realizations. See the book [BR84] for a good exposition. There is interesting recent work of C. Beck [B01] and results of Joe Ball, Tanit Malakorn, and Gilbert Groenewald [BGMprept]. Indeed, a very early version of the paper [BGMprept] provided our entry into the study of convexity for NC rational functions. The reader interested only in the broad outline of the paper need read only the first part of this section.

Section 5 is a brief section about NC directional derivatives and explains the connection between convexity and positive semidefinite second directional derivatives.

Corollary 6.1 at the outset of Section 6 provides enough background to understand the proof in Section 7 which shows that, under the hypothesis of convexity, symmetric minimal descriptor realizations (for scalar NC symmetric rational functions) are either monic, or have a J term with exactly one negative eigenvalue and are pinned. In the latter case, the realization can be unpinned to produce a pure monic butterfly realization as described in Section 8. This completes the proof of the realization conclusion of Theorem 3.3.

At this point the reader may wish to proceed to the short Section 13 which contains the proof of the growth conclusions of Theorem 3.3.

The singularities conclusion for descriptor realizations is proved in Section 9. This proof contains many of the ideas, in a somewhat cleaner form, needed for the proof of the singularities conclusion for pure butterfly realizations. It also explains the care taken in defining the domain of a NC rational function.

The proof of the singularities conclusion for butterfly realizations is spread over four sections. The first of these, Section 10 treats a Nullstellensatz for linear pencils. This section is short and sweet, potentially of independent interest, and can be read now. Section 11 uses the full strength of the results from Section 6 to reduce the problem to a situation close to that found in Section 9. The proof is completed in Section 12 which appeals to a real algebraic geometry result from Appendix 17.

Section 15 provides some motivation from the point of view of linear systems theory and can be read at any time.

Most sections from this point on start with a proposition which gives the main accomplishment of that section. Separate sections are typically devoted to different techniques, so a glance at the lead proposition will tell the reader if he wishes to read the section.

3.4. **Thanks.** The authors thank John Farina, Nick Slinglend and John Shopple for comments on the notes which lead to this manuscript. Also computer experiments by Adrian Lim, Brett Kotschwar, and Jeff Oval helped suggest our results and steer our proofs.

We are also grateful to the authors of [BGMprept] for sharing a very early version of their manuscript. It contributed greatly to the present paper.

We are grateful to Leonid Gurvits for bringing the results of Valiant on determinantal representations to our attention. We obtained our determinantal representation before knowing that Valiant's existed.

4. Realizations of r

This section begins with a review of the classical theory of descriptor realizations for NC rational functions tailored to future needs. See the book [BR84] for a more complete exposition and the papers [B01] [BGMprept] for recent developments. From the existence of descriptor realizations, a natural argument shows that symmetric NC rational functions have symmetric descriptor realizations. The section finishes with uniqueness results for symmetric descriptor and butterfly realizations. Thus the reader who is interested only in the descriptor case and is willing to accept later claims of uniqueness need read only the first two subsections.

4.1. **Descriptor Realizations.** Define a **descriptor realization** of a $d_1 \times d_2$ matrix NC rational function $\mathfrak{r}$ to be a rational expression

$$(x) = D + C(J - L_A(x))^{-1}B \tag{4.1}$$

for $\mathfrak{r}$, where $A_j \in \mathbb{R}^{d\times d}$ for $j = 1, \dots, g$, $D \in \mathbb{R}^{d_1\times d_2}$, $C \in \mathbb{R}^{d_1\times d}$ and $B \in \mathbb{R}^{d\times d_2}$. Here J denotes a $d \times d$ signature matrix, namely, $J = J^{\mathrm{T}}$ and $J^2 = I$. We emphasize that at this point the A_j are not required to be symmetric. Note that we could write equation (4.1) as

$$(x) = D + C(I - JL_A(x))^{-1}JB. \tag{4.2}$$

A **symmetric descriptor realization** is a descriptor realization with

$$D = D^{\mathrm{T}}, \quad B = C^{\mathrm{T}} \quad \textit{and the } A_j \textit{ are symmetric matrices.}$$

Clearly, the rational function $\mathfrak{r}$ corresponding to a symmetric descriptor realization is a symmetric rational function.

A descriptor realization is called **monic** provided $J = I$. It is **pinned** (resp. **unpinned**) if it uses a pencil which is pinned (resp. unpinned).

4.1.1. *Properties.* A descriptor realization is **observable** provided

$$C(JA)^w v = 0$$

for all words w implies $v = 0$. Similarly, it is **controllable** if

$$\mathrm{span}\,\{\mathrm{Range}\ (JA)^w JB : \textit{all words } w \textit{ in } \mathcal{W}_g\}$$

is all of $\mathbb{R}^d$. Since observability can also be expressed as the span of the ranges of $\{((JA)^w)^{\mathrm{T}} C^{\mathrm{T}}\}$ is all of $\mathbb{R}^d$ and $(A^T J)^w C^T = (JA^T)^w JC^T$ observability and controllability are the same for symmetric descriptor realizations.

We say that the descriptor realization is **minimal** if it is both observable and controllable. Because we wish to work with descriptor realizations with not necessarily 0 feed through term D, we have chosen this notion of minimality which differs slightly from asking that the state space (the space that the A_j and J act upon) have minimum dimension. These latter realizations are observable and controllable; however, there exists observable and controllable (as defined here) descriptor realizations for which the state space does not have minimum degree. Indeed, while we will not have explicit use for it, the interested reader should be able to prove, after reading this section, that any two minimal pinned (resp. unpinned) descriptor realizations for the same scalar NC rational function have the same state space dimension and that the dimension of an pinned realization is one larger than that of an unpinned realization. Thus, the difference in dimensions between any two minimal realizations is at most one. The calculations at the outset of §8.2 illustrates the passage of a minimal pinned realization to a minimal unpinned realization with a drop of one in the dimension of the corresponding state spaces. Keep in mind that there are NC rational expressions for which every descriptor realization is pinned.

Two minimal monic descriptor realizations with the same feed through term D,

$$\begin{aligned} &= D + C(I - L_A(x))^{-1}B \\ \tilde{} &= D + \tilde{C}(I - L_{\tilde{A}}(x))^{-1}\tilde{B}, \end{aligned}$$

for the same rational function are **similar** provided there exists an invertible matrix S such that

$$SA_j = \tilde{A}_j S, \quad SB = \tilde{B}, \quad C = \tilde{C}S.$$

The S is known as a **similarity transform**.

We say that **a realization is pinned** provided it uses a pencil which is pinned. We note for (resp. symmetric) descriptor realizations, pinned means that the A_j (or equivalently the JA_j) for $j = 1, \dots, g$ have a common null space.

We shall often substitute $n \times n$ matrices $X_1, \dots, X_g$ for $x_1, \dots, x_g$ in rational expressions such as (x) as discussed in §2.3 with specific formulas given in (2.4).

4.2. **Symmetric Descriptor Realizations Exist.** That noncommutative rational functions analytic at 0 have descriptor realizations can be found in [BR84]. Moreover, any two minimal descriptor realizations with the same feed through term D are similar. We now exploit, in a canonical and totally unoriginal way, the symmetry implicit in a symmetric rational function to show, by appropriate choice of similarity transform, that any symmetric noncommutative rational function $\mathfrak{r}$ analytic at 0 has a symmetric minimal descriptor realization; i.e., a symmetric descriptor realization which is minimal amongst all descriptor realizations.

Lemma 4.1. (1) *Any descriptor realization is (more precisely, determines) an NC matrix valued rational function which is analytic at* 0. *Conversely, each $m_1 \times m_2$ matrix valued NC rational function $\mathfrak{r}$ analytic at 0, has a minimal descriptor realization (which could be taken to be monic) with* 0 *feed through term ($D = 0$).*

Moreover, any two minimal descriptor realizations for $\mathfrak{r}$ with the same feed through term are similar via a unique similarity transform.

(2) *Any NC matrix valued rational function analytic at* 0 *with a symmetric descriptor realization is a symmetric rational function.*

(3) *If $\mathfrak{r}$ is a symmetric matrix valued NC rational function analytic at* 0, *then it has a minimal symmetric descriptor realization.*

(4) *If $\mathfrak{r}$ is a $m_1 \times m_2$ NC rational function analytic at* 0, *then minimality of the descriptor realization implies the pinning space of the realization has dimension at most m_2. In particular, if $\mathfrak{r}$ is scalar (1×1), then the pinning space has dimension at most one.*

Proof of Lemma 4.1 (1): This is a classical theorem due to Schutzenberger [S61].

The equivalence of formula (4.1) and formula (4.2) establishes the monic claim.

The uniqueness of the similarity transform is explicitly stated in [BGMprept], although it is implicit in the other references above. This statement is the **state space similarity theorem**. Later in this section when we discuss existence and uniqueness of butterfly realizations where there is no reference to cite we essentially copy the proof of the state space similarity theorem for FM realizations found in [BGMprept]. Indeed, it is possible to reduce the proof of the state space similarity theorem for butterfly realizations to that for FM realizations.

□

Proof of Lemma 4.1 (2): Obvious. □

Proof of Lemma 4.1 (3): See §4.3. □

Proof of Lemma 4.1 (4): Minimality implies

$$\mathbb{R}^d = \mathcal{S} + \ \textit{Range } JB$$

where

$$\mathcal{S} := \ \mathrm{span}\{\mathrm{Range}\ (JA)^w JB : \ \textit{all words } w \neq \ \textit{empty word}\}$$

and since *Range* JB has rank at most m_2, we see that $\mathcal{S}$ has codimension at most m_2. On the other hand, if γ pins A (meaning $A_j^{\mathrm{T}}\gamma = 0$ for all j), then γ is orthogonal to $J\mathcal{S}$. Thus, the dimension of the pinning space is at most the codimension of $\mathcal{S}$. □

4.3. From Descriptor to Symmetric: Proof of Lemma 4.1 item (3).

The proof of Lemma 4.1 item (3) is based on a construction which we shall summarize in the following lemma.

Lemma 4.2. *Suppose the monic descriptor realization*

$$(x) = D + C(I - L_A(x))^{-1}B \tag{4.3}$$

for a symmetric NC scalar rational function $\mathfrak{r}$ *is minimal and* D *is symmetric* ($D = D^{\mathrm{T}}$)*, but the* A_j *are not necessarily symmetric. Then there exists a unique invertible symmetric* $d \times d$ *matrix* S *such that*

$$SA_jS^{-1} = A_j^{\mathrm{T}}, \qquad SB = C^{\mathrm{T}} \qquad j = 1, \dots, g. \tag{4.4}$$

From this a symmetric minimal realization of $\mathfrak{r}$ *is obtained as follows:*

Factor S *as* $S = RJR^{\mathrm{T}}$ *where* J *is a* $d \times d$ *signature matrix and* R *is* ($d \times d$ *and) invertible. Set*

$$\widetilde{C} := C(R^{-1})^{\mathrm{T}} \quad \textit{and} \quad \widetilde{A}_j := JR^{\mathrm{T}}A_j(R^{-1})^{\mathrm{T}}. \tag{4.5}$$

Then

$$\tilde{}(x) = D + \widetilde{C}(J - L_{\widetilde{A}}(x))^{-1}\widetilde{C}^{\mathrm{T}} \tag{4.6}$$

is a minimal symmetric realization for $\mathfrak{r}$*. Moreover, the pencil* $L_{\widetilde{A}}(x)$ *is unpinned if and only if the pencil* $L_A(x)$ *is unpinned.*

Proof of Lemma 4.2 equation (4.4) Since $\mathfrak{r}$ is symmetric, the monic descriptor realization

$$^{T} = D + B^{\mathrm{T}}(I - L_{A^{\mathrm{T}}}(x))^{-1}C^{\mathrm{T}}$$

is also a minimal descriptor realization for $\mathfrak{r}$ with the same feed through term as . Hence, by the state space similarity theorem, there exists a unique invertible S with $SA_j = A_j^T S$, $SB = C^T$, and $C = B^T S$. Taking transposes of these three relations gives $S^T A_j = A_j^T S^T$, $C^T = B^T S^T$, and $BS^T = C^T$. Thus S^T is also a similarity transform and thus by uniqueness, $S = S^T$.

Proof of Lemma 4.2 equation (4.5) (4.6)

We first verify that the $\widetilde{A}_j$ are symmetric for $j = 1, \ldots, g$. The point is that $\widetilde{A}_j^{\mathrm{T}} = \widetilde{A}_j$ is equivalent to

$$(R^{-1})A_j^{\mathrm{T}} RJ = [JR^{\mathrm{T}} A_j (R^{-1})^{\mathrm{T}}]^{\mathrm{T}} = \widetilde{A}_j^{\mathrm{T}} = \widetilde{A}_j = JR^{\mathrm{T}} A_j (R^{-1})^{\mathrm{T}}.$$

Multiply by R and R^{T} to get this equals $A_j^{\mathrm{T}} RJR^{\mathrm{T}} = RJR^{\mathrm{T}} A_j$ which is $A_j^{\mathrm{T}} S = SA_j$ which we already obtained above.

Since

$$J - L_{\widetilde{A}}(x) = JR^T (I - L_A(x))(R^{-1})^T$$

we find

$$(I - L_A(x))^{-1} = (R^{-1})^{\mathrm{T}} (J - L_{\widetilde{A}}(x))^{-1} JR^{\mathrm{T}}. \tag{4.7}$$

Substituting equation (4.7) into (4.3) which we recall is

$$(x) = D + C(I - L_A(x))^{-1} B \tag{4.8}$$

gives

$$\begin{aligned} (x) = {} & D + C(R^{-1})^{\mathrm{T}} (J - L_{\widetilde{A}}(x))^{-1} JR^{\mathrm{T}} B \\ = {} & D + C(R^{-1})^{\mathrm{T}} (J - L_{\widetilde{A}}(x))^{-1} R^{-1} RJR^{\mathrm{T}} B \\ = {} & D + \widetilde{C}(J - L_{\widetilde{A}}(x))^{-1} R^{-1} SB \\ = {} & D + \widetilde{C}(J - L_{\widetilde{A}}(x))^{-1} R^{-1} C^{\mathrm{T}} \\ = {} & D + \widetilde{C}(J - L_{\widetilde{A}}(x))^{-1} \widetilde{C}^{\mathrm{T}} = \tilde{\ }(x) \end{aligned} \tag{4.9}$$

which gives the symmetric realization as desired. Note that, strictly speaking, the 's from equations (4.8) and (4.9) are not the same; however they are equivalent rational expressions and both represent $\mathfrak{r}$.

Finally we show minimality of our symmetric representation for $\mathfrak{r}$. The definitions $\widetilde{C} := CR^{-T}$ and $(J\widetilde{A}) := R^{\mathrm{T}} A (R^{-1})^{\mathrm{T}}$ of the symmetric systems imply $(J\widetilde{A})^w = R^{\mathrm{T}} A^w (R^{-1})^{\mathrm{T}}$ and

$$\widetilde{C}(J\widetilde{A})^w v = CA^w (R^{-1})^{\mathrm{T}} v$$

for all words including the empty word. So A, C observable is equivalent to $\widetilde{A}, \widetilde{C}$ observable. Controllability and observability are equivalent for a symmetric system. Also since R and J are invertible, the descriptor system unpinned is equivalent to the symmetric descriptor system being unpinned. □

4.4. **Uniqueness of Symmetric Descriptor Realizations.** There is a useful refinement of the state space similarity theorem for symmetric descriptor realizations.

Proposition 4.3. *If*

$$= D + C(J - L_A(x))^{-1} C^{\mathrm{T}} \qquad \textit{and} \qquad \tilde{\ } = D + \widetilde{C}(\widetilde{J} - L_{\widetilde{A}}(x))^{-1} \widetilde{C}^{\mathrm{T}}$$

are both symmetric descriptor realizations for the same NC scalar rational function (with the same symmetric feed through term D) and if S is the unique similarity transform relating the two realizations (which by Lemma 4.2 is symmetric), then $S\widetilde{J}S = J$.

Thus, if $J = I$, then $\widetilde{J} = I$ too and S is unitary. In particular any two monic ($J = I$) symmetric minimal descriptor realizations with the same feed through term for the same rational function are unitarily equivalent.

Proof. Since both and ˜ represent the same rational function (and share the feed through term D),

$$C(JA)^w JC^{\mathrm{T}} = \widetilde{C}(\widetilde{J}\widetilde{A})^w \widetilde{J}\widetilde{C}^{\mathrm{T}}$$

for all words w.

The invertible matrix S from the state space similarity theorem satisfies $SJA_j = \widetilde{J}\widetilde{A}_j S$, and $SJC^T = \widetilde{J}\widetilde{C}^T$. Hence, $SJA^\alpha JC = \widetilde{J}\widetilde{A}^\alpha \widetilde{J}\widetilde{C}$ for all words α.

Since the A_j and $\widetilde{A}_j$ are symmetric, it follows that

$$CJ(AJ)^{\beta^T} S^T \widetilde{J} S (JA)^\alpha JC^T = \widetilde{C}\widetilde{J}(\widetilde{A}\widetilde{J})^{\beta^T}(J\widetilde{A})^\alpha \widetilde{J}\widetilde{C}^T.$$

Combining equations (4.4) and (4.4) and the minimality of the realizations shows $S^T\widetilde{J}S = J$. $\square$

4.5. **Uniqueness of Butterfly Realizations.** Parallel to what we saw for descriptor realizations, there is a certain amount of uniqueness built into minimal butterfly realization which is needed in the proof of Proposition 3.4.

4.5.1. *Pure Butterfly Realizations.* Suppose that the NC scalar rational function $\mathfrak{r}$ has two minimal unpinned pure butterfly realizations,

$$(x) = {}_0 + {}_1(x) + \ell(x)\ell(x)^{\mathrm{T}} + \Lambda(x)(J - L_A(x))^{-1}\Lambda(x)^{\mathrm{T}}$$

and

$$\tilde{}(x) = \tilde{}_0 + \tilde{}_1(x) + \tilde{\ell}(x)\tilde{\ell}(x)^{\mathrm{T}} + \tilde{\Lambda}(x)(\tilde{J} - L_{\tilde{A}}(x))^{-1}\tilde{\Lambda}(x)^{\mathrm{T}},$$

where both A and $\tilde{A}$ are tuples of symmetric matrices of size $d \times d$ and size $\tilde{d} \times \tilde{d}$ respectively, Λ_j^{T} and $\tilde{\Lambda}_j^{\mathrm{T}}$ are in $\mathbb{R}^d$ and $\mathbb{R}^{\tilde{d}}$ respectively, and where both J and $\tilde{J}$ are signature matrices, so that $J = J^{\mathrm{T}} = J^{-1}$ and similarly for $\tilde{J}$.

Proposition 4.4. *The subset*

$$\mathcal{S} := \text{ span of } \{(JA)^w J\Lambda_j^{\mathrm{T}} : w \text{ is a nonempty word and } j = 1, \dots, g\}$$

of $\mathbb{R}^d$ is all of $\mathbb{R}^d$.

The dimensions of the state space of both realizations are the same; that is, $d = \tilde{d}$ and there is a $d \times d$ invertible matrix S satisfying

$$S(JA)^w = (\tilde{J}\tilde{A})^w S \text{ for every word } w \text{ and } S^{\mathrm{T}}\tilde{J}S = J. \tag{4.10}$$

Furthermore,

$$S(JA)^w J\Lambda_j^{\mathrm{T}} = (\tilde{J}\tilde{A})^w \tilde{J}\tilde{\Lambda}_j^{\mathrm{T}} \tag{4.11}$$

for every nonempty word w and $j = 1, \dots, g$.

The proof below is essentially that of the state space similarity theorem for FM-realizations as found in [BGMprept] and this proposition can be reduced to results there.

Proof. We reiterate that pure means $\Lambda_0 = \tilde{\Lambda}_0 = 0$. Suppose $\gamma \in \mathbb{R}^d$ is orthogonal to $\mathcal{S}$ in which case

$$0 = \gamma^{\mathrm{T}}(JA)^w J\Lambda_j^{\mathrm{T}}$$

for every nonempty word w and $j = 1, \dots, g$. Hence, for each m, j and word u,

$$0 = (\gamma^{\mathrm{T}} JA_m)(JA)^u J\Lambda_j^{\mathrm{T}}.$$

It follows from minimality that $0 = A_m^{\mathrm{T}} J^{\mathrm{T}} \gamma = A_m(J\gamma)$ for each m. The unpinned hypothesis now implies that $J\gamma = 0$. Thus $\gamma = 0$ and the first part of the proposition, $\mathcal{S} = \mathbb{R}^d$ is proved.

Comparing coefficients of $x^{\chi_m w \chi_j}$ in the power series expansions of the two representations for $\mathfrak{r}$ gives, for all nonempty[3] words w and all $m, j = 1, \dots, g$,

$$\Lambda_m (JA)^w J\Lambda_j^{\mathrm{T}} = \tilde{\Lambda}_m (\tilde{J}\tilde{A})^w \tilde{J}\tilde{\Lambda}_j^{\mathrm{T}}.$$

Consider

$$\gamma := \sum_{w,j} \gamma_{w,j}(JA)^w J\Lambda_j^{\mathrm{T}} \quad \textit{and} \quad \tilde{\gamma} := \sum_{w,j} \gamma_{w,j}(\tilde{J}\tilde{A})^w \tilde{J}\tilde{\Lambda}_j^{\mathrm{T}}.$$

where only finitely many terms in the sum are nonzero and the empty word does not appear. We shall show that $\gamma = 0$ implies $\tilde{\gamma} = 0$. For each $m = 1, \dots, g$ and word u we find,

$$\begin{aligned}
0 =& \Lambda_m (JA)^u \gamma \\
=& \sum_{w,j} \Lambda_m (JA)^u \gamma_{w,j} (JA)^w J\Lambda_j^{\mathrm{T}} \\
=& \sum_{w,j} \tilde{\Lambda}_m (\tilde{J}\tilde{A})^u \gamma_{w,j} (\tilde{J}\tilde{A})^w \tilde{J}\tilde{\Lambda}_j^{\mathrm{T}} \\
=& \tilde{\Lambda}_m (\tilde{J}\tilde{A})^u \left(\sum_{w,j} \gamma_{w,j} (\tilde{J}\tilde{A})^w \tilde{J}\tilde{\Lambda}_j^{\mathrm{T}} \right) \\
=& \tilde{\Lambda}_m (\tilde{J}\tilde{A})^u \tilde{\gamma}.
\end{aligned}$$

Consequently, the minimality of the $\tilde{}$ representation implies $\tilde{\gamma} = 0$.

The above allows us to define the linear mapping S by $S\gamma = \tilde{\gamma}$, that is,

$$S \sum_{w,j} \gamma_{w,j}(JA)^w J\Lambda_j^{\mathrm{T}} = \sum_{w,j} \gamma_{w,j}(\tilde{J}\tilde{A})^w \tilde{J}\tilde{\Lambda}_j^{\mathrm{T}}$$

(where again the sum is finite and w is not allowed to be the empty word) gives a well defined onto mapping $S : \mathbb{R}^d \to \mathbb{R}^d$. The computation above which shows that S is well defined also shows, by reversing the roles of A and

[3] for w empty we get $\Lambda_m J\Lambda_j^{\mathrm{T}} + \ell_m \ell_j^{\mathrm{T}} = \tilde{\Lambda}_m \tilde{J}\tilde{\Lambda}_j^{\mathrm{T}} + \tilde{\ell}_m \tilde{\ell}_j^{\mathrm{T}}$ for all m, j which is not too helpful.

$\tilde{A}$, that S is one to one. The definition of S implies conclusion (4.11) of the proposition.

Now we prove the formulas (4.10). Since

$$S(JA)^u(JA)^w J\Lambda_j^{\mathrm{T}} = (\tilde{J}\tilde{A})^u S(JA)^w J\Lambda_j^{\mathrm{T}}$$

for w not the empty word, we have $S(JA)^u = (\tilde{J}\tilde{A})^u S$.

Finally, we have, (for both u and w not the empty word)

$$\begin{aligned}((JA)^u J\Lambda_m^{\mathrm{T}})^{\mathrm{T}} S^{\mathrm{T}}\tilde{J}S(JA)^w J\Lambda_j^{\mathrm{T}} =&((\tilde{J}\tilde{A})^u\tilde{J}\tilde{\Lambda}_m^{\mathrm{T}})^{\mathrm{T}}\tilde{J}(\tilde{J}\tilde{A})^w\tilde{J}\tilde{\Lambda}_j^{\mathrm{T}}\\ =&(\tilde{\Lambda}_m(\tilde{J}\tilde{A})^{u^{\mathrm{T}}})(\tilde{J}\tilde{A})^w\tilde{J}\tilde{\Lambda}_j^{\mathrm{T}}\\ =&(\Lambda_m(JA)^{u^{\mathrm{T}}})(JA)^w J\Lambda_j^{\mathrm{T}}\\ =&((JA)^u J\Lambda_m^{\mathrm{T}})^{\mathrm{T}} J(JA)^w J\Lambda_j^{\mathrm{T}}.\end{aligned}$$

The equality above plus the first part of the proposition now imply $S^{\mathrm{T}}\tilde{J}S = J$. ☐

4.5.2. *Monic Pure Butterfly Realizations.* As a brief follow up to the last section, observe that in the special case $J = \tilde{J} = I$, the invertible mapping S is unitary, $S^{\mathrm{T}}S = I$. This is analogous to Proposition 4.3 and proves the first part of the following proposition.

Proposition 4.5. *Any two pure minimal unpinned butterfly realizations with $J = I$ which represent the same NC scalar rational function are unitarily equivalent.*

Any two pure minimal butterfly realizations (unpinned or not) with $J = I$ which represent the same NC scalar rational function have the same formal domain.

Proof. To prove the second part of the proposition, suppose

$$(x) = {}_0 + {}_1(x) + \ell(x)\ell(x)^{\mathrm{T}} + \Lambda(x)(I - L_A(x))^{-1}\Lambda(x)^{\mathrm{T}}$$

is a pure minimal butterfly realization which is perhaps pinned. What is used here is that the pinning space reduces not only A but also $J = I$. Thus, decomposing

$$A_j = \begin{pmatrix} 0 & 0 \\ 0 & \check{A}_j \end{pmatrix} \qquad \mathit{and} \qquad \Lambda_j = \begin{pmatrix} \hat{\Lambda}_j \\ \check{\Lambda}_j \end{pmatrix}$$

gives a new rational expression

$$\check{}(x) = {}_0 + {}_1(x) + (\ell(x)\ell(x)^{\mathrm{T}} + \hat{\Lambda}(x)\hat{\Lambda}(x)^{\mathrm{T}}) + \check{\Lambda}(x)(I - L_{\check{A}}(x))^{-1}\check{\Lambda}(x)^{\mathrm{T}}$$

equivalent to . The formal domains of and $\check{}$ consists of those $X \in \mathbb{S}^g$ for which $I - L_A(X)$ and $I - L_{\check{A}}(X)$ are invertible, respectively. Since

$$I - L_A(X) = I \oplus (I - L_{\check{A}}(X)),$$

these are the same. The $\check{}$ representation is a pure minimal unpinned representation. The second part of the proposition now follows from the first by unpinning both representations as needed. ☐

5. The Directional Derivative of a Realization

In this paper we shall use directional derivatives of NC matrix valued functions and rational expressions. The key fact, discussed in Subsection 5.2 at the end of the section, is that convexity corresponds to a positive semidefinite second derivative.

A first derivative $r'(x)[h]$ and a second derivative $r''(x)[h]$, of an NC matrix valued rational expression r with respect to x in direction h, where $h = (h_1, \dots, h_g)$ is an additional g-tuple of noncommuting indeterminates, are defined recursively from the rules

(1) If $p(x) = p_\emptyset + \sum_{j=1}^{g} p_j x_j$ is a NC matrix valued polynomial of degree at most one, then $p'(x)[h] = \sum_{j=1}^{g} p_j h_j$.
(2) The sum rule: if $r = r_1 + r_2$, then $r'(x)[h] = r_1'(x)[h] + r_2'(x)[h]$.
(3) The product rule: if $r = r_1 r_2$, then $r'(x) = r_1'(x)[h]r_2(x) + r_1(x)r_2'(x)[h]$.
(4) If r is the inverse $r = f^{-1}$ of a NC matrix valued rational expression f satisfying $f(0) \neq 0$, then $r'(x)[h] := -f^{-1}(x)f'(x)[h]f^{-1}(x)$.

This derivative and the second directional derivative discussed below behave exactly as expected on polynomials. For instance, with $p(x) = x^3$, one has $p'(x)[h] = x^2h + xhx + hx^2$ and $p''(x)[h] = 2xh^2 + 2h^2x + 2hxh$. For more on this see [CHSY03].

Applying these rules (in a natural order) to r, a NC rational expression analytic at 0, gives a new NC rational expression $r'(x)[h]$. These differentiation rules correspond to the natural differentiation rules on formal NC power series and as a consequence every rational expression r in the NC rational function equivalence class $\mathfrak{r}$ has derivative $r'(x)[h]$ which is a rational expression in the same equivalence class; of course we denote this new equivalence class by $\mathfrak{r}'$ and call it the **directional derivative of the rational function $\mathfrak{r}$ in direction** h. Also if the rules are applied to r in two different orders the resulting expressions are in the same equivalence class; this is a direct consequence of the corresponding property on power series.

The $\mathbb{SR}^{nm_1 \times nm_2}$ -valued rational function in $g(n+1)n/2$ variables $\mathtt{r}$ induced on $(\mathbb{SR}^{n\times n})^g$ by a NC $m_1 \times m_2$ matrix valued rational expression r has a directional derivative at X in direction H, $\frac{d\mathtt{r}}{dt}(X+tH)|_{t=0}$ which is the same function on $(\mathbb{SR}^{n\times n})^g \times (\mathbb{SR}^{n\times n})^g$ as $r'(X)[H]$, that is

$$r'(X)[H] = \frac{d\mathtt{r}}{dt}(X+tH)|_{t=0}. \tag{5.1}$$

This is true because the constructions behind r' and $\frac{d\mathtt{r}}{dt}(X+tH)|_{t=0}$ agree on polynomials of degree at most one and have the same recursion laws.

Likewise one can use recursion rules to define a second directional derivative $r''(x)[h]$, called **the Hessian of $\mathfrak{r}$**. Namely,

(1) $p''(x)[h] = 0$ if p is a NC matrix valued polynomial of degree at most one;
(2) if $r = r_1 + r_2$, then $r''(x)[h] = r_1''(x)[h] + r_2''(x)[h]$.

(3) if $r = r_1 r_2$, then

$$r''(x)[h] = r_1''(x)[h]r_2(x) + r_1'(x)[h]r_2'(x)[h] + r_1'(x)[h]r_2'(x)[h] + r_1(x)r_2''(x)[h];$$

(4)

$$(r^{-1})''(x)[h] = r^{-1}(x)[-r''(x)[h] + 2r^{-1}(x)r'(x)[h]r^{-1}(x)r'(x)[h]]r^{-1}(x).$$

Also second derivative of r coincides with the second directional derivative of the corresponding function $\mathtt{r}$ analogous to equation (5.1):

$$r''(X)[H] = \frac{d^2\mathtt{r}}{dt^2}(X + tH)|_{t=0}. \tag{5.2}$$

A side remark is that NCAlgebra computes derivatives quite effectively using the differentiation rules.

5.1. **Derivatives of Descriptor Realizations.** We now take derivatives of a symmetric function in terms of its symmetric descriptor realization $(x) = D + C(J - L_A(x))^{-1}C^{\mathrm{T}}$.

The first derivative of in direction h is

$$'(x)[x] = C(J - L_A(x)\ )^{-1}L_A[h](J - L_A(x)\ )^{-1}C^{\mathrm{T}}$$

where we used the fact $L_A'(x)[h[= L_A(h)$. The second derivative is

(5.3)

$$''(x)[h] := 2C\ (J - L_A(x)\ )^{-1}L_A[h](J - L_A(x)\ )^{-1}L_A[h](J - L_A(x)\ )^{-1}C^{\mathrm{T}}.$$

5.2. **Second Derivative Characterization of Convexity.** Central to the argument of this paper is the following Proposition.

Proposition 5.1. *The matrix convexity of the NC matrix valued rational expression r on a open set $\mathcal{D}^n$ in $(\mathbb{SR}^{n\times n})^g$ is equivalent to $r''(X)[H]$ being positive semidefinite at all $X \in (\mathbb{SR}^{n\times n})^g$ in $\mathcal{D}^n$ and all $H \in (\mathbb{SR}^{n\times n})^g$.*

This is shown in [HMer98] for scalar NC rational expressions when $\mathcal{D}^n$ is $(\mathbb{SR}^{n\times n})^g$ for all n. The proof easily extends to open convex sets $\mathcal{D}^n$ and matrix valued expressions. What is essential in this paper is only the scalar NC rational case.

Example 5.2. **Hessian Positive Semidefinite at 0, but not near 0** An NC symmetric polynomial can have $p''(0)[H] \succ 0$ for all H, but yet there exists X arbitrarily close to 0 such that $p''(X)[H]$ is not positive semidefinite.

Indeed, let $p(x) = x^2 + x^3$ and verify

$$p''(X)[H] = H^2 + XH^2 + HXH + H^2X$$

which for $X = 0$ is H^2 which is positive semidefinite for all H. Choose

$$X = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and compute $p''(X)[H] := \begin{pmatrix} 1 & t \\ t & 0 \end{pmatrix}$ which is not positive semidefinite for $t \neq 0$. $\square$

6. Linear Dependence of Symbolic Functions

The main result in this section, Theorem 6.6 says roughly that if a collection of rational functions has the property that they evaluate in a linearly dependent way on a collection of matrices (satisfying certain hypothesis), then there is a universal dependence relation. The full strength of the theorem is needed in Section 11 to deal with the singularities conclusion in Theorem 3.3 for butterfly realizations. However, we begin with an easily stated consequence of Theorem 6.6 which is sufficient to establish the existence of a monic realization statement and to prove the singularities statement for descriptor realizations in Theorem 3.3. In this section and henceforth in the paper we concentrate on scalar rational expressions and functions, unless explicitly stated otherwise.

Corollary 6.1. *Suppose $G_1, \dots, G_\ell$ and $\rho_1, \dots, \rho_s$ are rational expressions with ρ_j being symmetric and suppose for each X in the strict positivity domain*

$$\mathcal{D}^\rho := \{X \in \mathbb{S}^g : \quad \rho_1(X) \succ 0, \dots, \rho_s(X) \succ 0\}$$

the matrices $\{G_j(X) : j = 1, 2, \dots, \ell\}$ exist (are finite). If, for each $X \in \mathcal{D}^\rho$, the set $\{G_j(X) : j = 1, 2, \dots, \ell\}$ is linearly dependent; i.e., there exists a nonzero $\lambda(X) \in \mathbb{R}^\ell$ satisfying

$$0 = \sum_1^\ell \lambda(X)_j G_j(X),$$

then there exists a nonzero $\lambda \in \mathbb{R}^\ell$ such that

$$0 = \sum_1^\ell \lambda_j G_j(X)$$

for all $X \in \mathcal{D}^\rho$, i.e., this λ is independent of $X \in \mathcal{D}^\rho$.

Before proving it we shall introduce some terminology pursuant to our more general result.

6.1. **Direct Sums.**

We present some definitions about direct sum and sets which respect direct sums, since they are important tools.

Definition 6.2. Our definition of the **direct sum** is the usual one. Given pairs (Ω, v_1) and (Ξ, v_2) where Ω, Ξ are $n_j \times n_j$ matrices and $v_j \in \mathbb{R}^{n_j}$,

$$(\Omega, v_1) \oplus (\Xi, v_2) = (\Omega \oplus \Xi, v_1 \oplus v_2)$$

where

$$\Omega \oplus \Xi := \begin{pmatrix} \Omega & 0 \\ 0 & \Xi \end{pmatrix} \qquad v_1 \oplus v_2 := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

The direct sum of matrix g-tupple X and vector u with matrix g-tuple Y and vector v is the matrix g-tuple

$$(X_1 \oplus Y_1, \cdots, X_g \oplus Y_g)$$

together with the vector $u \oplus v$. We denote this direct sum of g-tuples by $(X \oplus Y, u \oplus v)$. Extend these definitions for μ terms in the expected way.

In the definition below, we consider a set $\mathcal{B}$ which is the union

$$\mathcal{B} := \cup_{n=1}^{\infty} \mathcal{B}^n,$$

where each $\mathcal{B}^n$ is a set whose members are pairs (X, v) where X is in $(\mathbb{SR}^{n\times n})^g$ and $v \in \mathbb{R}^n$.

Definition 6.3. The **set $\mathcal{B}$ is said to respect direct sums** if $(X(i), v(i))$ with $X(i) \in (\mathbb{SR}^{n_j\times n_j})^g$ and $v(i) \in \mathbb{R}^{n_j}$ for $i = 1, \dots, \mu$ is contained in the set $\mathcal{B}$ implies that the direct sum

$$(X(1) \oplus \dots \oplus X(\mu), v(1) \oplus \dots \oplus v(\mu)) = (\oplus_{j=1}^{\mu} X(j), \oplus_{j=1}^{\mu} v(j))$$

is also contained in $\mathcal{B}$. As a special case, we could take $X(j) = X$ for $j = 1, 2, \dots, \mu$ in which case we find $(X^\mu, v^\mu) \in \mathcal{B}$, where X^μ denotes the direct sum of X with itself μ times and likewise for v^μ.

Definition 6.4. By a **natural map** G on $\mathcal{B}$, we mean a sequence of functions $G(n) : \mathcal{B}^n \to \mathbb{R}^n$, which **respects direct sums** in the sense that, if $(X^j, v^j) \in \mathcal{B}^{n_j}$ for $j = 1, 2, \dots, \mu$, then

$$G(\sum_1^{\mu} n_j)(\oplus X_j, \oplus v_j) = \oplus_1^{\mu} G(n_j)(X_j, v_j).$$

Typically we omit the argument n, writing $G(X)$ instead of $G(n)(X)$.

Examples of sets which respect direct sums and of natural maps are provided by the following lemma.

Lemma 6.5. *Given rational expressions $\rho_1, \dots, \rho_s$ consider the strict positivity domain*

$$\mathcal{D}^\rho := \{X \in \mathbb{S}^g \cap (\cap_j^s \mathcal{F}_{\rho_j, for}) : \quad \rho_1(X) \succ 0, \dots, \rho_s(X) \succ 0\}.$$

1. *The set $\mathcal{B}(\rho) = \{(X, v) : X \in \mathcal{D}^\rho \cap (\mathbb{SR}^{n\times n})^g,\ v \in \mathbb{R}^n \quad for\ some\ n\}$ respects direct sums.*
2. *If G is a matrix-valued NC rational expression whose domain contains $\mathcal{D}^\rho$, then G determines a natural map on $\mathcal{B}(\rho)$ by $G(n)(X, v) = G(X)v$.*

Proof of Lemma 6.5 Obvious. ☐

6.2. **Main Result on Linear Dependence: Uncontrollability.**

Theorem 6.6. *Suppose $\mathcal{B}$ is a set which respects direct sums and $G_1^i, \dots, G_{\ell_i}^i$ are natural maps on $\mathcal{B}$ where $i \in \mathcal{I}$ for some finite index set $\mathcal{I}$. If for each $(X, v) \in \mathcal{B}$ there exists an $i = i(X, v) \in \mathcal{I}$ such that the set $\{G_j^i(X, v) : j = 1, 2, \dots, \ell_i\}$ is linearly dependent, then there exists $d \in \mathcal{I}$ and a nonzero $\lambda \in \mathbb{R}^{\ell_d}$ so that*

$$0 = \sum_{j=1}^{\ell_d} \lambda_j G_j^d(X, v) \tag{6.1}$$

for every $(X, v) \in \mathcal{B}$. Once again, we emphasize that d and λ are independent of $(X, v) \in \mathcal{B}$.

Proof of Corollary 6.1 Using the notation and hypothesis of Corollary 6.1, let $\mathcal{B}$ denote the set

$$\mathcal{B}^n = \{(X,v) : \quad X \in \mathcal{D}^\rho \textit{ and } v \in \mathbb{R}^n\}.$$

Let G_j denote the natural maps, $G_j(X,v) = G_j(X)v$. Then $\mathcal{B}$ and $G_1, \dots, G_\ell$ satisfy the hypothesis of Theorem 6.6 and so the conclusion of Corollary 6.1 follows. $\square$

6.3. **Proof of Theorem 6.6.** The following lemma is a finite version of Theorem 6.6; namely, that given a finite subset $\mathcal{S} \subset \mathcal{B}$ one can find a d and nonzero λ which solves equation (6.1) independent of $(X,v) \in \mathcal{S}$.

Lemma 6.7. *Let $\mathcal{B}$ be our set that respects direct sums and let G^i_j denote our natural maps on $\mathcal{B}$ which satisfy the hypothesis of Theorem 6.6. If $\mathcal{S}$ is a finite subset of $\mathcal{B}$, then there exists $\mathrm{d}(\mathcal{S}) \in \mathcal{I}$ and a nonzero $\lambda(\mathcal{S}) \in \mathbb{R}^{\ell_{d(\mathcal{S})}}$ such that*

$$\sum_{j=1}^{\ell_{d(\mathcal{S})}} \lambda(\mathcal{S})_j G_j^{d(\mathcal{S})}(X)v = 0, \tag{6.2}$$

for every $(X,v) \in \mathcal{S}$.

Proof The proof relies on taking direct sums of matrices. Write the set $\mathcal{S}$ as $\mathcal{S} = \{(X^1,v^1), \dots, (X^\mu, v^\mu)\}$, where each $(X^i, v^i) \in \mathcal{B}$ for $i = 1, \dots, \mu$. Since $\mathcal{B}$ respects direct sums,

$$(X,v) = (\oplus_{\nu=1}^{\mu} X^\nu, \oplus_{\nu=1}^{\mu} v^\nu)$$

is in $\mathcal{B}$. Hence, there exists $d(\mathcal{S}) \in \mathcal{I}$ and $\lambda(\mathcal{S}) \in \mathbb{R}^{\ell_{d(\mathcal{S})}}$ such that

$$0 = \sum_{j=1}^{\ell_{d(\mathcal{S})}} \lambda(\mathcal{S})_j G_j^{d(\mathcal{S})}(X,v). \tag{6.3}$$

Since each G^i_j respects direct sums, it follows that

$$0 = \oplus_{\nu=1}^{\mu} \sum_{j=1}^{\ell_{d(\mathcal{S})}} \lambda(\mathcal{S})_j G_j^{d(\mathcal{S})}(X^\nu, v^\nu) \tag{6.4}$$

from which it follows that

$$0 = \sum_{j=1}^{\ell_{d(\mathcal{S})}} \lambda(\mathcal{S})_j G_j^{d(\mathcal{S})}(X^\nu, v^\nu)$$

for each $\nu = 1, 2, \dots, \mu$. $\square$

Proof of Theorem 6.6. For $i \in \mathcal{I}$, let $\mathbb{B}_i$ **denote the closed unit sphere** in $\mathbb{R}^{\ell_i}$. (Beware $\mathbb{B}$ is not the ball.) We view $\mathbb{B}_i$ as a the subset of $\oplus_{j \in \mathcal{I}} \mathbb{B}_j$ given the inclusion

$$\gamma \in \mathbb{B}_i \mapsto 0 \oplus \cdots \oplus 0 \oplus \gamma \oplus 0 \cdots \oplus 0$$

where γ is in the i-th coordinate. Let $\mathbb{B}$ denote the union of the $\mathbb{B}_i$ viewed as included in the direct sum. In particular, with $k = \sum_{i \in \mathcal{I}} \ell_i$, the unit sphere $\mathbb{B}_k$ in $\mathbb{R}^k$ contains $\mathbb{B}$.

Given $\lambda \in \mathbb{B}$, write $\lambda = \oplus_{i\in\mathcal{I}}\lambda^i$, that is,

$$\lambda^i = \begin{pmatrix} \lambda^i_1 \\ \vdots \\ \lambda^i_{\ell_i} \end{pmatrix},$$

and note that there is a μ such that $\|\lambda^\mu\| = 1$ and $\lambda^\nu_\ell = 0$ for $\nu \neq \mu$. For such λ and $(X, v) \in \mathcal{B}$, let

$$\lambda \cdot G(X)v = \sum_{\nu\in\mathcal{I}} \sum_{j=1} \lambda^\nu_j G^\nu_j(X, v)$$

To $(X, v) \in \mathcal{B}$ associate the set

$$\Omega_{(X,v)} = \{\lambda \in \mathbb{B} : \lambda \cdot G(X)v = 0\}.$$

Since $(X, v) \in \mathcal{B}$, the hypothesis on $\mathcal{B}$ says $\Omega_{(X,v)}$ contains λ making

$$0 = \sum_{j=1} \lambda^i_j G^i_j(X, v)$$

for some i and λ can be rescaled to be in $\mathbb{B}_i \subset \mathbb{B}$. It is evident that $\Omega_{(X,v)}$ is a closed subset of $\mathbb{B}$ and is thus compact.

Let $\boldsymbol{\Omega}$ denote the collection $\{\Omega_{(X,v)} : (X, v) \in \mathcal{B}\}$ of subsets of $\mathbb{B}$. Any finite sub-collection from $\boldsymbol{\Omega}$ has the form $\{\Omega_{(X,v)} : (X, v) \in \mathcal{S}\}$ for some finite subset $\mathcal{S}$ of $\mathcal{B}$, and so by Lemma 6.7 has a nonempty intersection. In other words, $\boldsymbol{\Omega}$ has the finite intersection property. The compactness of $\mathbb{B}$ implies that there is a $\lambda \in \mathbb{B}$ which is in every $\Omega_{(X,v)}$. This is the conclusion of the theorem. $\square$

7. Convexity of r Plus Unpinned Implies J is Positive Definite

In this section we analyze the effect that convexity of a scalar rational function $\mathfrak{r}$ forces on its symmetric minimal descriptor realizations. The following is the main result of the section.

Proposition 7.1. *Suppose*

$$(x) = D + C(J - L_A(x))^{-1}C^{\mathrm{T}}.$$

is a minimal symmetric descriptor realization of the scalar rational function $\mathfrak{r}$. *Assume there exists an* $\epsilon > 0$ *such that*

$$\frac{1}{2}\ ''(X)[H] = C(J - L_A(X))^{-1}L_A(H)(J - L_A(X))^{-1}L_A(H)(J - L_A(X))^{-1}C^{\mathrm{T}}$$

is analytic and positive semidefinite for all X *and* H *in* $(\mathbb{SR}^{n\times n})^g$ *satisfying* $X_1^2 + \cdots + X_g^2 \prec \varepsilon I$.

1. *If the symmetric pencil* $J - L_A(X)$ *is unpinned, then* J *is positive definite. Thus, without loss of generality, we can take* $J = I$.
2. *Define* $\alpha_0 \in \mathbb{R}^{d\times gd}$ *to be the matrix whose block form is*

$$\alpha_0 := \begin{pmatrix} A_1 & A_2 & \ldots A_g \end{pmatrix}$$

and let $P_{\alpha_0} \in \mathbb{R}^{d\times d}$ *denote the orthogonal projection onto the range of* α_0. *If the realization is pinned, then* $P_{\alpha_0} J P_{\alpha_0}$ *is positive semidefinite. Moreover since* $\mathfrak{r}$ *is a scalar NC rational function the pinning space has* *dimension at most one (by Lemma 4.1 item 4), so the codimension of Range α_0 is at most one. This implies that J has at most one negative eigenvalue.*

Conversely, the formula for $''$ shows that it is positive semidefinite where the pencil is positive definite. If J is positive definite this includes the origin, so $''$ is positive on the set $\{X \in \mathbb{S}^g : \ J - L_A(X) \succ 0\}$.

The proof is an instructive guide to more complicated proofs to come, which overlap this one. It requires three lemmas. Define Γ to be the d-dimensional vector valued NC rational expression

$$\Gamma(x) := (J - L_A(x))^{-1} C^{\mathrm{T}}. \tag{7.1}$$

Let $\mathfrak{d}$ denote the rational function determined by Γ. $(J - L_A(x))$ is a matrix rational function and thus so are the entries $e_j^{\mathrm{T}}(J - L_A(x))^{-1} C^{\mathrm{T}}$ of $\mathfrak{d}$.

Recall, Γ is evaluated at a tuple of matrices $X \in (\mathbb{S}\mathbb{R}^{n \times n})^g$ for which $J \otimes I - L_A(X)$ is invertible via the formula

$$\Gamma(X) := (J \otimes I_n - L_A(X))^{-1} (C^{\mathrm{T}} \otimes I_n).$$

Lemma 7.2. *Given a symmetric pencil $J - L_A(x)$ invertible on $\{X \in \mathbb{S}^g : X_1^2 + \cdots + X_g^2 \prec \varepsilon I\}$.*

EITHER

(1) *There is an $X \in (\mathbb{S}\mathbb{R}^{n \times n})^g$ satisfying*

$$X_1^2 + \cdots + X_g^2 \prec \varepsilon I \tag{7.2}$$

and a v in $\mathbb{R}^n$ such that the vector z in $\mathbb{R}^{nd}$ defined by

$$z := \Gamma(X) v \in \mathbb{R}^{nd} = \mathbb{R}^n \otimes \mathbb{R}^d$$

has components $z^1, \dots, z^d$ which are linearly independent vectors in $\mathbb{R}^n$

OR

(2) *There is a vector λ in $\mathbb{R}^d$ such that*

$$\sum_j^d \lambda_j \mathfrak{d}_j = 0. \tag{7.3}$$

Remark 7.3. Condition (2) is equivalent to saying that the rational function determined by $\sum \lambda_j \Gamma_j$ is equivalent to 0. Note also, that this is a purely algebraic condition on the coordinate functions Γ_j of Γ; no test matrices appear.

Lemma 7.4. *Condition (2) in Lemma 7.2 violates controllability of the realization of Γ in (7.1), that is, of $JC^{\mathrm{T}}, A_j J$. Note that the unpinned hypothesis is not needed.*

For an $n \times n$ matrix M, define $diag_M = \ I_d \otimes M$.

Lemma 7.5. *Suppose the symmetric pencil $J, A_j,\ j = 1, \dots g$ is unpinned. Partition $z \in \mathbb{R}^{nd}$ as $(z^1, \dots, z^d)$ where each z^j is in $\mathbb{R}^n$. Suppose the vectors $(z^1, \dots, z^d)$ are linearly independent. The subspace*

$$\{L_A(H)z : \ \text{all } H_j = H_j^{\mathrm{T}}\} \tag{7.4}$$

which equals

$$\{\ (A_1 \otimes I_n) diag_{H_1} z + \ldots + (A_g \otimes I_n) diag_{H_g} z : \quad \text{all } H_1 = H_1^{\mathrm{T}}, \dots, H_g = H_g^{\mathrm{T}}\}$$

has codimension at most $\frac{d(d-1)}{2} g$ in $\mathbb{R}^{nd}$ independent of how large n is.

If the pencil is pinned or unpinned, then the subspace $\{L_A(H)z : \ \text{all } H_j = H_j^{\mathrm{T}}\}$ has codimension at most $\frac{d(d-1)}{2} g$ in the range of $P_{\alpha_0} \otimes I_n$ independent of how large n is.

Proof Lemma 7.4: The transpose of equation (7.3) says that the vector λ satisfies $0 = C(J - L_A(x))^{-1}\lambda$. This is equivalent to $0 = CJ(AJ)^w\lambda = C(JA)^w J\lambda$ for all words w which, since the realization is observable, implies that $0 = J\lambda$, so $\lambda = 0$. □

Proof of Lemma 7.2: In the hypothesis of Corollary 6.1 take

1. Take ρ_1 to be $\rho_1 := \varepsilon - x_1^2 + \cdots + x_g^2$ and $\rho_2 := 0, \dots, \rho_s := 0$.
2. Take $G_j(x) := \Gamma(x)_j$ for $j = 1, \dots, d$.

Thus, if we assume that condition Lemma 7.2 does not hold, then we can apply the corollary to conclude there is a $\lambda \in \mathbb{R}^d$ in such that

$$h(x) := \sum_j \lambda_j \Gamma(x)_j$$

satisfies $h(X) = 0$ for any g-tuple X of symmetric matrices in $\mathbb{R}^{n \times n}$, satisfying $X_1^2 + \cdots + X_g^2 \preceq \varepsilon I$.

It follows from Lemma 2.2 that the equivalence class determined by the rational expression h is 0. This is the conclusion of the lemma. □

Proof of Lemma 7.5 : The $d \times dg$ matrix

$$\alpha_0 = \begin{pmatrix} A_1 & A_2 & \dots A_g \end{pmatrix}$$

has range equal to $\mathbb{R}^d$ provided the pencil J, A_j is unpinned. It follows that the $nd \times ndg$ matrix

$$\begin{aligned} \alpha &= \begin{pmatrix} A_1 \otimes I_n & A_2 \ \otimes I_n & \dots A_g \otimes I_n \end{pmatrix} \\ &= \begin{pmatrix} A_1 & A_2 \dots A_g \end{pmatrix} \otimes I_n \\ &= \alpha_0 \otimes I_n \end{aligned}$$

has range equal to $\mathbb{R}^{nd}$ provided the pencil J, A_j is unpinned. Below we view α as a map $\alpha : \oplus_1^g \mathbb{R}^{nd} \to \mathbb{R}^{nd}$.

Lemma 9.5 [CHSY03] says (since the z^j are linearly independent) for each $j = 1, \dots, g$, the subspace

$$\mathcal{S}_j := \{diag_{H_j} z : \ all \ H_j = H_j^{\mathrm{T}}\} = \{ \begin{pmatrix} H_j z^1 \\ \vdots \\ H_j z_d \end{pmatrix} : all \ H_j = H_j^{\mathrm{T}}\} \subset \mathbb{R}^{ng} = \mathbb{R}^{nd}$$

of $\mathbb{R}^{nd}$ has codimension no greater than $\frac{d(d-1)}{2}$. Thus the subspace $\mathcal{S} := \mathcal{S}_1 \oplus \dots \oplus \mathcal{S}_g$ of $\oplus_1^g \mathbb{R}^{nd}$ has codimension $\le \frac{d(d-1)}{2} g$. Since α is onto, it follows that the codimension of $\alpha\mathcal{S}$ in $\mathbb{R}^{nd}$ is at most $\frac{d(d-1)}{2} g$.

When the system is pinned, α is not onto. However, the argument above gives that the codimension of $\{L_A(H)z : \ all \ H_j = H_j^{\mathrm{T}}\}$ in Range α is bounded by $\frac{d(d-1)}{2} g$. □

Proof of Proposition 7.1:

Because of controllability, Lemma 7.2 says we can pick X^*, v^* such that $z^* := \Gamma(X^*)v^*$ has components $z^{*1}, \dots, z^{*d}$ which are linearly independent vectors in $\mathbb{R}^n$ and so that $X_1^2 + \cdots + X_g^2 \prec \varepsilon I$. By convexity,

$$\begin{aligned} 0 &\le v^{*T}\Gamma(X^*)^{\mathrm{T}} L_A(H)(J \otimes I - L_A(X^*))^{-1} L_A(H)\Gamma(X^*)v^* \\ &= z^{*T} L_A(H)(J - L_A(X^*))^{-1} L_A(H) z^* \end{aligned} \tag{7.5}$$

for all H.

Suppose $(J \otimes I - L_A(X^*))^{-1}$ is not positive semidefinite; then direct sum X^* with itself $\frac{d(d-1)}{2} g + 1$ times to obtain the matrix X^{**} acting on $\mathbb{R}^{n(\frac{d(d-1)}{2} g+1)}$. Then $(J \otimes I - L_A(X^*))^{-1}$ has at least $\frac{d(d-1)}{2} g + 1$ negative eigenvalues. The d components of the corresponding vector z^{**} (the direct sum of z^* with itself $\frac{d(d-1)}{2} g + 1$ times) are $z^{*1} \oplus \dots \oplus z^{*1}, \dots, z^{*d} \oplus \dots \oplus z^{*d}$ which are linear independent. Plug X^{**} and the corresponding z^{**} into (7.5) and from it get that

$$\operatorname{codim}\{L_A(H) z^{**} : all \ H_j^{\mathrm{T}} = H_j\} > \frac{d(d-1)}{2} g.$$

Thus Lemma 7.5 pertains and implies in the unpinned case that this codimension is larger than required, so we have a contradiction. Hence $J \otimes I - L_A(X^*)$ is positive semidefinite.

When the system is pinned the argument above gives

$$P_{\alpha_0}(J \otimes I - L_A(X^*))^{-1} P_{\alpha_0}$$

is positive semidefinite. This tells us we can take $\varepsilon \to 0$ and obtain $X_\varepsilon^* \to 0$ satisfying

$$P_{\alpha_0}(J \otimes I - L_A(X_\varepsilon^*))^{-1} P_{\alpha_0}$$

is positive semidefinite. Item (2) of Proposition 7.1 now follows since

$$P_{\alpha_0}(J \otimes I - L_A(X_\varepsilon^*))^{-1} P_{\alpha_0} \to P_{\alpha_0} J \otimes I P_{\alpha_0}.$$

When the system is unpinned $P_{\alpha_0} = I$, so J is positive semidefinite, which of course implies it is I. □

8. Unpinning the Descriptor Realization

This section treats a symmetric minimal descriptor realization

$$(x) = C(J - L_A(x))^{-1}C^{\mathrm{T}},$$

for a scalar noncommutative rational function. In particular, J is a signature matrix and the A_j are symmetric. Proposition 8.1 gives an algorithm to pass from a symmetric descriptor realization which is pinned and for which J has at most one negative eigenvalue to an unpinned realization, either symmetric descriptor or butterfly. This is the main result in this section. Before proving the Proposition, it is used to deduce the realization conclusion of Theorem 3.3. The discussion ends with a formulation of the construction here as an algorithm suitable for computer implementation, see Section 8.4. The final subsection gives the proof of the easy direction of the Convexity Region conclusion of Theorem 3.3.

Throughout this section $\mathfrak{r}$ is a scalar symmetric NC rational function; and the scalar hypothesis is seriously used (for the first time in our proofs).

Proposition 8.1. *If*

$$(x) = C(J - L_A(x))^{-1}C^{\mathrm{T}},$$

is a minimal symmetric descriptor realization (which has therefore at most one pinning vector by Lemma 4.1 item 4) and the signature matrix J has exactly one or no negative eigenvalue, then **EITHER** *the NC rational function $\mathfrak{r}$ it represents has*

(1) *a minimal unpinned symmetric descriptor realization*

$$\tilde{}(x) = \widetilde{D} + \widetilde{C}(J_0 - L_{\widetilde{A}}(x))^{-1}\widetilde{C}^{\mathrm{T}}$$

where J_0 is a signature matrix with either one or no negative eigenvalues. We emphasize that $\widetilde{A}$ is symmetric;

OR

(2) *$\tilde{\mathfrak{r}}$ has a minimal unpinned butterfly realization,*

$$(x) = {}_0 + {}_1(x) + \ell(x)\ell(x)^{\mathrm{T}} + \Lambda(x)(I - L_{\widetilde{A}}(x))^{-1}\Lambda(x)^{\mathrm{T}},$$

as defined in (3.2). We emphasize that $J = I$ and it is possible that some of the terms, including the last, may be absent.

8.1. **Convexity: Proof of Realization in Theorem 3.3.**

Suppose $\mathfrak{r}$ is in $\mathbb{R}\langle x\rangle_{Rat0}$ which is matrix convex near 0. We shall show it has an unpinned minimal butterfly realization as in (3.5) with $J = I$.

That $\mathfrak{r}$ is an NC symmetric rational function implies that it has a minimal (though possibly pinned) descriptor realization (with the feedthrough term D equal to zero) by Lemma 4.1. That $\mathfrak{r}$ has a minimal descriptor realization and is NC matrix convex near 0 implies that the signature matrix J in the descriptor realization has at most one negative eigenvalue and the pinning space $\{\gamma : A_j\gamma = 0 \text{ all } j = 1, \dots, g\}$ is at most one dimensional by Proposition 7.1 item 2 which is precisely the hypothesis of Proposition 8.1. An application of Proposition 8.1 gives either a monic butterfly realization which is the conclusion of the realization part of Theorem 3.3, or an unpinned minimal descriptor realization. In the latter case, Proposition 7.1 item 1 implies that J is positive definite, so without loss of generality $J = I$.

The pure butterfly realization part of Theorem 3.3 follows from Lemma 8.2 below.

8.2. **Proof of Proposition 8.1.** Without loss of generality, assume that $e_1 := (1, 0, 0, \dots)^{\mathrm{T}} \in \mathbb{R}^d$ is the pinning vector, so that

$$A_j = \begin{pmatrix} 0 & 0 \\ 0 & \widehat{A}_j \end{pmatrix}.$$

Since, by hypothesis, the pinning space has dimension at most one, the common null space of the matrices $\widehat{A_j}$ is trivial.

Decomposing the signature matrix J with respect to the orthogonal decomposition $\mathbb{R}^1 \oplus \mathbb{R}^{d-1}$ we have,

$$J = \begin{pmatrix} \alpha & \beta^{\mathrm{T}} \\ \beta & \delta \end{pmatrix}.$$

The condition $J^2 = I$, implies $\delta\beta = -\alpha\beta$, so that either $\beta = 0$, or β is an eigenvector for δ. We consider these two cases separately.

8.2.1. *Suppose* $\beta = 0$. Here there are two cases, $\alpha = 1$ and $\alpha = -1$.

If $\alpha = -1$, then

$$J = \begin{pmatrix} -1 & 0 \\ 0 & \delta \end{pmatrix}$$

but δ is a positive definite signature matrix, so $\delta = I$. and so, with

$$C = \begin{pmatrix} C_0 & C_1 \end{pmatrix},$$

where $C_0 \in \mathbb{R}^1$ and $C_1 \in \mathbb{R}^{d-1}$ we have,

$$(x) = -C_0^2 + C_1(I - L_{\widehat{A}}(x))^{-1}C_1^{\mathrm{T}}. \tag{8.1}$$

Next, suppose that $\alpha = 1$. In this case,

$$J = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix},$$

where δ is a signature matrix with at most one negative eigenvalue. We find,

$$(x) = C_0^2 + C_1(J_0 - L_{\widehat{A}}(x))^{-1}C_1^{\mathrm{T}}. \tag{8.2}$$

Note, in both (8.1) and (8.2), the realizations are unpinned. We also need to see they are minimal. But this is evident from the simple form of the realizations.

8.2.2. *Suppose* $\beta \neq 0$. Use the abbreviation L defined by

$$L_A(x) = \begin{pmatrix} 0 & 0 \\ 0 & \sum_j \widehat{A}_j x_j \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \widehat{L}(x) \end{pmatrix}$$

to define L a pencil which is symmetric and unpinned; note we do not make the x dependence of L explicit in the notation. Compute,

(8.3)

$$\begin{aligned} &= C(J - L_A(x))^{-1}C^{\mathrm{T}} = C(I - JL_A(x))^{-1}JC^{\mathrm{T}} \\ &= C\begin{pmatrix} I & \beta^{\mathrm{T}}\widehat{L}(I-\delta\widehat{L})^{-1} \\ 0 & (I-\delta\widehat{L})^{-1} \end{pmatrix} JC^{\mathrm{T}} \\ &= \begin{pmatrix} C_0 & C_1 \end{pmatrix} \begin{pmatrix} I & \beta^{\mathrm{T}}\widehat{L}(I-\delta\widehat{L})^{-1} \\ 0 & (I-\delta\widehat{L})^{-1} \end{pmatrix} \begin{pmatrix} \alpha C_0 + \beta^{\mathrm{T}} C_1^{\mathrm{T}} \\ \beta C_0 + \delta C_1^{\mathrm{T}} \end{pmatrix} \\ &= C_0(\alpha C_0 + \beta^{\mathrm{T}} C_1^{\mathrm{T}}) + (C_0\beta^{\mathrm{T}}\widehat{L} + C_1)(I-\delta\widehat{L})^{-1}(\beta C_0 + \delta C_1^{\mathrm{T}}). \end{aligned}$$

The analysis of the $\beta \neq 0$ case continues by considering the separate cases $\alpha = 0$ and $\alpha \neq 0$.

First, **suppose** $\alpha \neq \mathbf{0}$**.** In this case,

$$\beta^{\mathrm{T}} = -\frac{1}{\alpha}\beta^{\mathrm{T}}\delta \tag{8.4}$$

and both $\alpha^2 + \beta^T\beta = 1$ and $\delta^2 = I - \beta\beta^T$. These last two equalities imply that δ^2 is strictly positive and hence δ is invertible. Using equation (8.4),

$$\begin{aligned} C_0\beta^{\mathrm{T}}\widehat{L}(I-\delta\widehat{L})^{-1} &= -\frac{C_0}{\alpha}\beta^{\mathrm{T}}\delta\widehat{L}(I-\delta\widehat{L})^{-1} \\ &= \frac{C_0}{\alpha}\beta^{\mathrm{T}}(I-\delta\widehat{L})(I-\delta\widehat{L})^{-1} - \frac{C_0}{\alpha}\beta^{\mathrm{T}}(I-\delta\widehat{L})^{-1} \\ &= \frac{C_0}{\alpha}\beta^{\mathrm{T}} - \frac{C_0}{\alpha}\beta^{\mathrm{T}}(I-\delta\widehat{L})^{-1}. \end{aligned} \tag{8.5}$$

Substituting equation (8.5) into equation (8.3) gives,

$$= C_0(\alpha C_0 + \beta C_1^{\mathrm{T}}) + \frac{C_0}{\alpha}\beta^{\mathrm{T}}(\beta C_0 + \delta C_1^{\mathrm{T}}) + (-\frac{C_0}{\alpha}\beta^{\mathrm{T}} + C_1)(I-\delta\widehat{L})^{-1}(\beta C_0 + \delta C_1^{\mathrm{T}}).$$

(Strictly speaking the equalities above and below means equivalence of rational expressions.) Finally, using once again equation (8.4),

$$= \text{constant} + (-\frac{C_0}{\alpha}\beta^{\mathrm{T}} + C_1)(I-\delta\widehat{L})^{-1}\delta(-\frac{C_0}{\alpha}\beta + C_1^{\mathrm{T}}).$$

Thus,

$$= D + E^{\mathrm{T}}(I-\delta\widehat{L})^{-1}\delta E \tag{8.6}$$

where $E := -\frac{C_0}{\alpha}\beta + C_1^{\mathrm{T}}$.

Note that δ and $\widehat{L}$ are both symmetric, but of course $\delta\widehat{L}$ need not be. (On the plus side, δ is invertible, and $\widehat{L}$ is unpinned, so that $\delta\widehat{L}$ is unpinned.) Further while δ is symmetric δ need not be positive definite (depending on α). To fix these deficiencies, use the fact that δ is symmetric and invertible with at most one negative eigenvalue to choose a positive definite matrix Δ

and a signature matrix $\mathcal{J}$ with either one or no negative eigenvalues so that $\delta = \Delta\mathcal{J}\Delta$. Substitution into equation (8.6) produces,

$$= D + E\Delta(\mathcal{J} - \Delta\widehat{L}\Delta)^{-1}\Delta E^{\mathrm{T}} \tag{8.7}$$

which is a symmetric realization. (Since Δ is symmetric, so are the matrices $\Delta\widehat{A}_j\Delta$.)

Since Δ is invertible, the matrices $\Delta\widehat{A}_j\Delta$ have a nontrivial common null space if and only if the matrices $\widehat{A}_j$ have a nontrivial common null space. Hence the representation in equation (8.7) is unpinned.

We now turn to minimality. Straightforward computation reveals

$$(JA)^{\mu}JC^{\mathrm{T}} = \begin{pmatrix} * \\ (\delta\widehat{A})^{\mu}(\beta C_0 + \delta C_1^{\mathrm{T}}) \end{pmatrix}.$$

Hence, as the original realization was minimal, the span of

$$\{(\delta\widehat{A})^{\mu}(\beta C_0 + \delta C_1^{\mathrm{T}}) : \ \textit{all words } \mu\}$$

is all of $\mathbb{R}^{d-1}$. Since also $\delta E = \beta C_0 + \delta C_1^{\mathrm{T}}$ and

$$(\mathcal{J}\Delta\widehat{A}\Delta)^{\mu}\mathcal{J}\Delta E = \Delta^{-1}(\delta\widehat{A})^{\mu}\delta E$$

it follows that the representation of equation (8.7) is minimal.

Finally, we take up **the case $\alpha = 0$.** Here we get $\delta\beta = 0$, so δ is not invertible. In this case, we may assume

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}. \tag{8.8}$$

where J is being decomposed as a map from $\mathbb{R}^1 \oplus \mathbb{R}^1 \oplus \mathbb{R}^{d-2}$ to itself. Also in these coordinates we decompose $\widehat{L}$ and C also

$$L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & L_{11} & L_{21}^{\mathrm{T}} \\ 0 & L_{21} & L_{22} \end{pmatrix}$$

$$C = \begin{pmatrix} C_0 & C_{11} & C_{12}^{\mathrm{T}}. \end{pmatrix}$$

We compute

$$(I - JL)^{-1} = \begin{pmatrix} 1 & L_{11} + L_{21}^{\mathrm{T}}(I - L_{22})^{-1}L_{21} & L_{21}^{\mathrm{T}}(I - L_{22})^{-1} \\ 0 & 1 & 0 \\ 0 & (I - L_{22})^{-1}L_{21} & (I - L_{22})^{-1} \end{pmatrix}$$

and

$$JC^{\mathrm{T}} = \begin{pmatrix} C_{11} \\ C_0 \\ C_{12} \end{pmatrix}.$$

Hence,

$$= (C_0C_{11} + C_{11}C_0) + C_0^2L_{11} + (C_0L_{21}^{\mathrm{T}} + C_{12}^{\mathrm{T}})(I - L_{22})^{-1}(C_0L_{21} + C_{12}). \tag{8.9}$$

Note that L_{22} could be pinned, but we emphasize that $J = I$. Because the pencil $I - L_A(x)$ in (8.9) is monic, it is straightforward to pass to a minimal realization by cutting the system down to

$$\textit{span Range}\ \{(\widehat{A}_{22})^{\mu}\Lambda_j^{\mathrm{T}} :\ \textit{all}\ \mu,\ j = 0, \dots, g\}$$

where $\Lambda_0 = C_{12}$, and $L_{21} = \sum \Lambda_j x_j$.

Summarizing, if we start with a pinned descriptor realization where J has exactly one negative eigenvalue, then we obtain one of the four realizations (8.1), (8.2), (8.7), or (8.9).

Now (8.1), (8.2), (8.7) are minimal unpinned descriptor realizations as required by conclusion (1) of Proposition 8.1. Realization (8.9) meets the requirements of (2) except that the realization may be pinned. Since $J = I$ in the realization, the pinning space reduces the pencil $J - L_A(x)$ thereby splitting its inverse into two pieces. One of these is an $\ell\ell^{\mathrm{T}}$ term and the other piece is an unpinned realization. Thus we get an unpinned butterfly realization with $J = I$ as required by the proposition. $\square$

8.3. **Producing a Pure Butterfly Realization.** We remark that the butterfly realization can be taken to have a slightly more restricted form. This is needed in several of our proofs.

Lemma 8.2. *If $\mathfrak{r}$ has an unpinned butterfly realization,*

$$(x) = {}_0 + {}_1(x) + \ell(x)\ell(x)^{\mathrm{T}} + \Lambda(x)(I - L_A(x))^{-1}\Lambda(x)^{\mathrm{T}}$$

where Λ is affine linear in x, then $\mathfrak{r}$ has a minimal unpinned pure butterfly realization,

$$(x) = \tilde{}_0 + \tilde{}_1(x) + \tilde{\ell}(x)\tilde{\ell}(x)^{\mathrm{T}} + L_{\widetilde{\Lambda}}(x)(I - L_{\widetilde{A}}(x))^{-1}L_{\widetilde{\Lambda}}(x)^{\mathrm{T}}$$

Here $L_{\widetilde{\Lambda}}$ is linear, that is, $L_{\widetilde{\Lambda}}(x) = \sum_1^g \widetilde{\Lambda}_j x_j$.

Indeed, more is true.

Lemma 8.3. *If $\mathfrak{r}$ has a butterfly realization,*

$$(x) = {}_0 + {}_1(x) + \ell(x)\ell(x)^{\mathrm{T}} + \Lambda(x)(J - L_A(x))^{-1}\Lambda(x)^{\mathrm{T}}$$

where Λ is affine linear in x, and J is a signature matrix, then $\mathfrak{r}$ has a pure butterfly realization, that is

$$(x) = \tilde{}_0 + \tilde{}_1(x) + \tilde{\ell}(x)\tilde{\ell}(x)^{\mathrm{T}} + L_{\widetilde{\Lambda}}(x)(J - L_{\widetilde{A}}(x))^{-1}L_{\widetilde{\Lambda}}(x)^{\mathrm{T}}$$

where $L_{\widetilde{\Lambda}}$ is linear. Beware the realization may be pinned.

Proof: The proof is constructive. For $j = 1, \dots, g$ let $\widetilde{\Lambda}_j = \Lambda_0 J A_j + \Lambda_j$ and let

$$L_{\widetilde{\Lambda}}(x) = \sum_{j=1}^{g} \widetilde{\Lambda}_j x_j.$$

denote the corresponding pencil. We have,

$$\Lambda(x) = \Lambda_0 J(J - L_A(x)) + L_{\widetilde{\Lambda}}(x).$$

Thus,

$$\begin{aligned}
&\Lambda(x)(J-L_A(x))^{-1}\Lambda(x)^{\mathrm{T}}\\
&\quad=\Lambda_0 J(J-L_A(x))(I-L_A(x))^{-1}(J-L_A(x))J\Lambda_0^{\mathrm{T}}\\
&\qquad+\Lambda_0 J(J-L_A(x))(J-L_A(x))^{-1}L_{\widetilde{\Lambda}}(x)^{\mathrm{T}}\\
&\qquad+L_{\widetilde{\Lambda}}(x)(J-L_A(x))^{-1}(J-L_A(x))J\Lambda_0^{\mathrm{T}}+L_{\widetilde{\Lambda}}(x)(J-L_A(x))^{-1}L_{\widetilde{\Lambda}}(x)^{\mathrm{T}}\\
&\quad=\Lambda_0 J(J-L_A(x))J\Lambda_0^{\mathrm{T}}+\Lambda_0 J L_{\widetilde{\Lambda}}(x)^{\mathrm{T}}\\
&\qquad+L_{\widetilde{\Lambda}}(x)J\Lambda_0^{\mathrm{T}}+L_{\widetilde{\Lambda}}(x)(J-L_A(x))^{-1}L_{\widetilde{\Lambda}}(x)^{\mathrm{T}}.
\end{aligned}$$

The first three terms in the left hand side are combinations of constant and linear terms and the last is as desired. □

Proof of Lemma 8.2 If we began with an unpinned observable and controllable representation, the alternate butterfly realization is unpinned, but it may not be minimal. However, because we are assuming $J=I$, restricting to the A reducing subspace spanned by $\{A^w\tilde{\Lambda}_j : w \text{ a word}, 1\le j\le g\}$ gives a minimal unpinned realization. □

8.4. **Convexity and Unpinning as an Algorithm.** In this subsection we briefly formulate the construction and result of the previous sections assuming that the (scalar) noncommutative symmetric rational function $\mathfrak{r}$ is convex near the origin, for someone whose primary interest is computer implementation in mind.

As mentioned in §3 there are algorithms which produce a minimal symmetric descriptor realization for a symmetric NC rational function $\mathfrak{r}$. The realization may be pinned and J may not be I.

Given a symmetric minimal descriptor realization

$$= C(J-L_A(x))^{-1}C^{\mathrm{T}} \tag{8.10}$$

of $\mathfrak{r}$ which is matrix convex near the origin[4], we shall produce such a realization with $J=I$ which is unpinned.

Proposition 7.1 implies the realization has at most a one dimensional pinning space and the signature matrix J has exactly one or no negative eigenvalue. Thus there are two cases.

If $J=I$, unpinning (if necessary) and cutting down to a symmetric minimal descriptor realization is straightforward. Conversely, if the representation is unpinned, then convexity implies $J=I$. Summarizing,

(1) either $J=I$ or equivalently the realization (8.10) is unpinned;
(2) or J has exactly one negative eigenvalue and the representation is pinned.

[4]That is, this descriptor realization satisfies the hypothesis of Proposition 8.1.

It is this second case that requires some effort. First change variables to make $\gamma = e_1$ a vector spanning the pinning space. Next put J into the form

$$J = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & -\alpha & 0 \\ 0 & 0 & I \end{pmatrix}.$$

for some α, β. The case that α and β are both nonzero leads to the minimal unpinned realization of equation (8.7); however, this is not compatible with $\mathfrak{r}$ convex near 0. Similarly, the case that $\alpha = 1, \quad \beta = 0$ also cannot occur. Thus, the only possible cases are $\alpha = -1, \quad \beta = 0$ and $\alpha = 0, \quad \beta = \pm 1$. The first case is trivially dealt with.

As for the second case, let

$$\hat{A}_j = PA_jP \quad \textit{and} \quad \Delta = PJP,$$

where P is the projection onto the span of $\{e_2, \dots, e_d\}$ (the orthogonal complement of the pinning vector). With the choices

$$C = \begin{pmatrix} C_0 & C_1 \end{pmatrix} \qquad \textit{and} \qquad \Lambda_j = (C_0^{\mathrm{T}} e_2 + \hat{A}_j \Delta C_1^{\mathrm{T}}),$$

we have

$$= CJC^{\mathrm{T}} + C(JAJ)C^{\mathrm{T}} + L_\Lambda(x)^{\mathrm{T}}(I - \Delta L_{\hat{A}}(x)\Delta)^{-1} L_\Lambda(x).$$

This realization is definitely pinned; however, since now $J = I$ we produce a minimal (consequently unpinned) realization as is standard by compression to the span of $\{\hat{A}^w \Lambda_j : \textit{ all words } w \}$. Compression produces a $J = I$ realization.

This completes the construction.

8.5. **The Region where $\mathfrak{r}$ is Convex Contains $\mathcal{P}$.** Now we prove the domain of convexity of the butterfly realization (3.5) with $J = I$ includes $\mathcal{P}$.

This follows from the formula for in Proposition 7.1 in the way that the last statement of Proposition 7.1 is proved. The fact is not new with us, c.f. C. K. Li and R. Mathais [LM00] which shows that the Schur complement

$$f(w, x, y) := w + x(I - y)^{-1} x^{\mathrm{T}}$$

is (jointly) matrix convex in the variables w, x, y on the set of matrices Y where $I - Y$ is positive definite. $\square$

9. Singularities in the Descriptor Realization

Up to this point we have proved that a convex NC scalar rational function $\mathfrak{r}$ has a minimal unpinned butterfly realization. The next four sections analyze the "poles" of $\mathfrak{r}$ versus the "zeroes" of the pencil inverted in the realization. (The reader who wants to skip this topic can also go to Section 13.) This section proves the correspondence in the descriptor case. The main result of this section, Proposition 9.1 is the singularities conclusion in Theorem 3.3 for monic descriptor realizations. The ideas and approach ultimately help to prove the Singularity conclusion of Theorem 3.3 for monic butterfly realizations but this is involved and requires three sections, §10, 11, 12.

Proposition 9.1. *Suppose* $\mathfrak{r}$ *is a rational function and the expression*

$$(x) = D + C(I - L_A(x))^{-1}C^{\mathrm{T}}. \tag{9.1}$$

is a minimal symmetric descriptor realization for $\mathfrak{r}$*. Then the domain of definition of* $\mathfrak{r}$ *and the formal domains of and of the rational expression* G *defined by* $G(x) := (I - L_A(x))^{-1}$ *all coincide, that is,*

$$\mathcal{F}_{\mathfrak{r}} = \mathcal{F}_{G,for} = \mathcal{F}_{,for}.$$

Consequently, $\mathcal{F}^0_{\mathfrak{r}} = \mathcal{F}^0_{G,for}$*.*

There is no advantage to adding the hypothesis that is unpinned in Proposition 9.1. To see this, decompose, with respect to the pinning space,

$$A = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{A} \end{pmatrix}, \quad C = \begin{pmatrix} C_0 & \tilde{C} \end{pmatrix}$$

let $\tilde{D} = D + C_0 C_0^{\mathrm{T}}$ and note that the realization

$$\tilde{\ } = \tilde{D} + \tilde{C}(I - L_{\tilde{A}}(x))^{-1}\tilde{C}^{\mathrm{T}}$$

is a minimal unpinned realization and $(I - L_A(x))^{-1}$ and $(I - L_{\tilde{A}(x)})^{-1}$ have the same domains.

From the representation (9.1) it is evident that $\mathcal{F}_{G,\mathrm{for}} = \mathcal{F}_{,\mathrm{for}} \subset \mathcal{F}_{\mathfrak{r}}$. To pursue the reverse inclusion suppose it is false, that is, suppose there is an $X \notin \mathcal{F}_{,\mathrm{for}}$ but which is in $\mathcal{F}_{\mathfrak{r}}$. Concretely, this means that

> *there is a rational expression* r *equivalent to such that* $X \in \mathcal{F}_{r,for}$*, but* $I - L_A(X)$ *is not invertible.*

We shall derive a contradiction with an argument very much like the proof of Proposition 7.1. We use r to denote throughout the proof a rational function meeting the conditions italicized above.

9.1. **Buried Singularities?** Define

$$\mathcal{N} = \{(X, w) : X \in (\mathbb{SR}^{n\times n})^g, \ w \neq 0, \ [I - L_A(X)]w = 0\}$$

and

$$\mathcal{N}_1 = \{X : \ (X, w) \in \mathcal{N} \text{ for some } w\}.$$

Thus, $\mathcal{N}_1$ is the complement of $\mathcal{F}_{,\mathrm{for}}$.

Let

$$\mathfrak{B}_r := \mathcal{N}_1 \cap \mathcal{F}_{r,\mathrm{for}}.$$

We call $\mathfrak{B}_r$ **the buried singularity set of the descriptor realization relative to the rational expression** r**,** a long name, often abbreviated buried singularity set. While the subscript r does not actually indicate which (this is not so important in view of Proposition 4.3), it does help distinguish one buried singularity set form another in some situations. Reiterating the discussion above, the conclusion of Proposition 9.1 is that $\mathfrak{B}_r$ is empty.

9.2. Γ **Redux.** Recall the abbreviation $\Gamma(x) := (J - L_A(x))^{-1}C^{\mathrm{T}}$; here in this section we take $J = I$. Note that $\mathcal{F}_{\Gamma,\mathrm{for}}$ is exactly the tuples X for which $I - L_A(X)$ is invertible. Thus $\mathcal{F}_{,\mathrm{for}} = \mathcal{F}_{\Gamma,\mathrm{for}}$. The next lemma extends the definition of $\Gamma(X)$ to $\mathfrak{B}_r$.

Because of the form of $I - L_A(X)$, there is $\varepsilon > 0$ such that if $X \in (\mathbb{SR}^{n\times n})^g$ and $X_1^2 + \cdots + X_g^2 \prec \varepsilon I$, then $(I - L_A(X))^{-1}$ exists independent of n so that X is in the formal domain of Γ. Of course this is true for all rational expressions. In particular, without loss of generality, if $X_1^2 + \cdots + X_g^2 \prec \varepsilon I$, then X is in the formal domain of r also.

Lemma 9.2. *Suppose X is g-tuple of $n \times n$ symmetric matrices.*

(1) *If $X \in \mathfrak{B}_r$, then*

$$\Gamma(X) := \lim_{t\to 1} \Gamma(tX) \in \mathbb{R}^d \otimes \mathbb{SR}^{n\times n}$$

exists.

In particular, if $X \in \mathcal{F}_{\Gamma,for}$, then, by continuity, this definition of $\Gamma(X)$ agrees with the original definition.

(2) *If $X_1^2 + \cdots + X_g^2 \prec \varepsilon I$, then*

$$\Gamma(X) = \lim_{t\to 1} \Gamma(tX).$$

(3) *If $Y \in \mathfrak{B}_r$ and $X_1^2 + \cdots + X_g^2 \prec \varepsilon I$, then $X \oplus Y \in \mathfrak{B}_r$ and*

$$\Gamma(X \oplus Y) = \Gamma(X) \oplus \Gamma(Y).$$

Proof. Let $\mathcal{K}$ denote the subspace $\ker(I - L_A(X))$ of $\mathbb{R}^{nd}$. Since $X \in \mathcal{N}_1$, this subspace is nontrivial, but that is not so important for the argument to follow. With respect to the orthogonal decomposition of $\mathbb{R}^{nd}$ as $\mathcal{K} \oplus \mathcal{K}^\perp$, we have

$$L_A(X) = \begin{pmatrix} I & 0 \\ 0 & I - P \end{pmatrix}$$

for some invertible P. We also decompose the $nd \times n$ matrix

$$C^{\mathrm{T}} \otimes I = \begin{pmatrix} (C^{\mathrm{T}})_1 I \\ (C^{\mathrm{T}})_2 I \\ \vdots \\ (C^{\mathrm{T}})_d I \end{pmatrix} \tag{9.2}$$

as

$$C^{\mathrm{T}} \otimes I = \begin{pmatrix} B^0 \\ B^1 \end{pmatrix}$$

with respect to orthogonal decomposition $\mathcal{K} \oplus \mathcal{K}^\perp$. In particular, as P is invertible, for t near 1, but $t \neq 1$, we get that $I - tL_A(X)$ is invertible.

The assumption $X \in \mathcal{F}_r$ means $r(tX)$ is defined for t near 1 (including $t = 1$)

$$
\begin{aligned}
&r(tX) = C(I - tL_A(X))^{-1} tL_A(X) C^{\mathrm{T}} \\
&= \begin{pmatrix} (B^0)^{\mathrm{T}} & (B^1)^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} (1-t)I & 0 \\ 0 & (1-t)I + tP \end{pmatrix}^{-1} \begin{pmatrix} tI & 0 \\ 0 & t(I-P) \end{pmatrix} \begin{pmatrix} B^0 \\ B^1 \end{pmatrix} \\
&= \begin{pmatrix} (B^0)^{\mathrm{T}} & (B^1)^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \frac{t}{1-t} I & 0 \\ 0 & ((1-t)I + tP)^{-1} t(I-P) \end{pmatrix} \begin{pmatrix} B^0 \\ B^1 \end{pmatrix},
\end{aligned}
$$

from which it immediately follows that $B^0 = 0$ and

$$
\lim_{t \to 1} \Gamma(tX) = \begin{pmatrix} 0 \\ P^{-1}(I-P)B^1 \end{pmatrix}.
$$

Lemma 9.2 (2) is evident.

Since $X_1^2 + \cdots + X_g^2 \prec \varepsilon I$, the tuple X is in the formal domain of r. Since also $Y \in \mathcal{F}_{r,\mathrm{for}}$, it follows from Proposition 2.1 that $X \oplus Y \in \mathcal{F}_{r,\mathrm{for}}$. Since $Y \in \mathcal{N}_1$ so is $X \oplus Y$. This shows $X \oplus Y \in \mathfrak{B}_r$. The last statement follows from the first two items. $\square$

9.3. **Finish the Proof of Proposition 9.1.** We now return to decomposing $C^{\mathrm{T}} \otimes I$ with respect to $\mathbb{R}^n \otimes \mathbb{R}^d$ as in (9.2) and correspondingly, given X in $(\mathbb{SR}^{n \times n})^g$ for which $\Gamma(X)$ is defined, write

$$
\Gamma(X) = \begin{pmatrix} \Gamma(X)_1 \\ \vdots \\ \Gamma(X)_d \end{pmatrix} \in \mathbb{R}^d \otimes \mathbb{SR}^{n \times n}
$$

By Lemmas 7.2 and 7.4 there is a tuple $\check{X}$ in $(\mathbb{SR}^{n \times n})^g$ and vector $\check{v}$ such that $\check{X}^{\mathrm{T}} \check{X} \prec \varepsilon I$ and the set of vectors $\{\check{z}_j = \Gamma(\check{X})_j \check{v}\}$ is linearly independent.

To finish the proof it suffices to show $\mathfrak{B}_r$ is empty. And this we argue by contradiction reasoning much as in Proposition 7.1. The details follow.

Choose $Y \in \mathfrak{B}_r$ and let $X^* = Y \oplus \check{X}$ and $v^* = 0 \oplus \check{v}$. The vectors

$$
\{z_j^* := \Gamma(X^*)_j v^* : j = 1, \ldots, d\}
$$

are linearly independent since their compressions $\check{z}_j$ to the second coordinates form a linearly independent set. Take X^{**} to be the $\frac{d(d-1)}{2} + 1$ fold direct sum of X^* with itself; take v^{**} and z^{**} to be the corresponding direct sums of v^* and z^*.

The remainder of the proof is summarized by the following slightly more general statement which is needed in Section 12.

Lemma 9.3. *If   is a minimal descriptor realization as in Proposition 9.1 which is also assumed to be unpinned, then there does not exist a tuple X^{**} of symmetric matrices satisfying all of the following*

(i) *the kernel of $I - L_A(X^{**})$ has dimension at least $\frac{d(d-1)}{2} + 1$;*
(ii) $\Gamma(X^{**}) = \lim_{t \to 1}(I - tL_A(X^{**}))^{-1} C^{\mathrm{T}}$ *exists;*
(iii) *for each tuple H, $''(tX^{**})[H]$ is bounded for $0 < t < 1$; and*

(iv) *there is a vector* v^{**} *with the property* $z^{**} = \Gamma(X^{**})v^{**}$ *has linearly independent components; i.e., writing* $z^{**} = \sum_1^d e_j \otimes z_j \in \mathbb{R}^d \otimes \mathbb{R}^n$, *the set* $\{z_1, \dots, z_d\} \subset \mathbb{R}^n$ *is linearly independent.*

Proof. For notational purposes let $N = n(\frac{d(d-1)}{2} + 1)$, the dimension of the space that $L_A(X^{**})$ acts on. We begin with a X^{**} in $(\mathbb{SR}^{n\times n})^g$. The dimension count produced by Lemma 7.5 tells us there is g-tuple H of symmetric matrices H_j in $\mathbb{R}^{N\times N}$ and a vector v^{**} such that $w := L_A(H)\Gamma(X^{**})v^{**}$ is a nonzero vector in $\ker(I - L_A(X^{**}))$.

Substituting tX^{**} and H into the formula (5.3) for second derivatives gives,

$$
\begin{aligned}
&\frac{1}{2} {v^{**}}^{\mathrm{T}} \; ''(tX^{**})[H]v^{**} \\
&\quad = {v^{**}}^{\mathrm{T}} \Gamma(tX^{**}) L_A(H)(I - tL_A(X^{**}))^{-1} L_A(H)\Gamma(tX^{**})v^{**}.
\end{aligned} \tag{9.3}
$$

We now decompose $\mathbb{R}^N$ into $\mathcal{K} \oplus \mathcal{K}^\perp$, where $\mathcal{K}$ is the kernel of $I - L_A(X^{**})$ as was done in the proof of Lemma 9.2. With respect to this decomposition,

$$
(I - tL_A(X^{**}))^{-1} = \begin{pmatrix} \frac{1}{1-t} I & 0 \\ 0 & ((1-t)I + tP)^{-1} \end{pmatrix} \tag{9.4}
$$

for some invertible matrix P and

$$
L_A(H)\Gamma(tX^{**})v^{**} = w(t) = \begin{pmatrix} w^0(t) \\ w^1(t) \end{pmatrix} \tag{9.5}
$$

where the $w^j(t)$ are continuous at 0 with limit (denoted $w^j(1)$) existing as $t \to 1$, by virtue of item (ii). Since $w(1) \in \mathcal{K}$, we have $w^1(1) = 0$. Substituting equations (9.4) and (9.5) into equation (9.3) gives,

$$
\begin{aligned}
\frac{1}{2} {v^{**}}^{T} \; (tX^{**})[H]v^{**} &= \begin{pmatrix} w^0(t)^{\mathrm{T}} & w^1(t)^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \frac{1}{1-t} I & 0 \\ 0 & ((1-t)I + tP)^{-1} \end{pmatrix} \begin{pmatrix} w^0(t) \\ w^1(t) \end{pmatrix} \\
&= \frac{1}{1-t} w^0(t)^{\mathrm{T}} w^0(t) + w^1(t)^{\mathrm{T}}((1-t)I + tP)^{-1} w^1(t),
\end{aligned}
$$

which goes to ∞ as t tends to 1 from below, since $w^0(1)$ is not zero. This contradicts item (iii) and completes the proof of the Lemma. □

Returning to the proof of Proposition 9.1, the fact that the X^{**} constructed before the statement of the Lemma is in $\mathcal{F}_{r,\mathrm{for}}$ implies that X^{**} satisfies item (iii) of the Lemma. This completes the proof of Proposition 9.1. □

We remark that the use of the second derivative of $\mathfrak{r}$ in this proof was a device to give variations in X. The convexity of $\mathfrak{r}$ near 0 was not used.

10. Homogeneous Pencils with Comparable Zero Sets; Nullpencilsatz

At this point we have proved Theorem 3.3 up through the Singularities statement for descriptor realizations. The proof of the singularities statement for pure butterfly realizations is somewhat more involved. Accordingly, it has been split into three parts, Sections 10, 11 and 12. The subject of this section, a nullstellensatz for linear pencils, may be of independent interest.

Suppose $\Omega = (\Omega_1, \dots, \Omega_g)$ with each $\Omega_j \in \mathbb{R}^{d\times d}$ and $\Lambda = (\Lambda_1, \dots, \Lambda_{\tilde g})$ with each $\Lambda_j \in \mathbb{R}^{k\times d}$ and let

$$L_\Omega(x) := \sum_1^g \Omega_j x_j \qquad L_\Lambda(x) := \sum_1^g \Lambda_j x_j$$

denote the corresponding pencils.

Proposition 10.1. *If there is an $n > g$ such that whenever $X \in (\mathbb{SR}^{n\times n})^g$ we have $L_\Omega(X)v = 0$ implies $L_\Lambda(X)v = 0$, then there is $k \times d$ matrix M satisfying*

$$\Lambda_j = M\Omega_j.$$

In particular,

$$L_\Lambda(x) = ML_\Omega(x).$$

The proof of Proposition 10.1 uses the following lemma.

Lemma 10.2. *Suppose $\mathcal{N}$, $\mathcal{E}$ and $\mathcal{F}$ are finite dimensional Hilbert spaces and $T : \mathcal{N} \to \mathcal{E}$ and $S : \mathcal{N} \to \mathcal{F}$ are linear maps. If ker$(T) \subset$ ker(S), then there exists a linear map $M : \mathcal{E} \to \mathcal{F}$ so that $S = MT$.*

Proof. Let $[T]$ denote the induced mapping

$$[T] : \mathcal{N}/\ker(T) \to \mathcal{E}.$$

Note that the kernel inclusion hypothesis implies that S induces a well defined linear map

$$[S] : \mathcal{N}/\ker(T) \to \mathcal{F}$$

given by $[S](h + \ker(T)) = Sh$. Let W denote the inverse of $[T]$ (with range restriction),

$$W : T(\mathcal{N}) \to \mathcal{N}/\ker(T)$$

This exists as everything is finite dimensional, $[T]$ is one-one and onto its range. Extend W to be zero on the orthocomplement of $T(\mathcal{N})$. Then

$$SWTh = SW[T]h = Sh.$$

Thus, choosing $M = SW$ proves the lemma. □

In the proof of the proposition it will be convenient to represent $L_\Omega(X)$ as a block matrix with block matrix entries. The tensor product of an $n \times n$ matrix A with an $m \times m$ matrix B gives rise to an operator on $\mathbb{R}^{nm}$ and thus can be represented as a matrix. Indeed, each choice of (ordered) orthonormal basis produces a representation. For instance, letting $\{e_1, \dots, e_n\}$ and $\{e'_1, \dots, e'_m\}$ denote the usual orthonormal bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively let $\{f_1, \dots, f_{mn}\}$ denote the (ordered) orthonormal basis of $\mathbb{R}^{mn}$ given by $f_{(j-1)m+k} = e_j \otimes e'_k$ for $1 \le j \le n$ and $1 \le k \le m$. Writing $B = (B_{k,\ell})$, with respect to this basis $A \otimes B$ has the block matrix representation,

$$\begin{pmatrix} AB_{1,1} & AB_{1,2} & \dots & AB_{1,m} \\ AB_{2,1} & AB_{2,2} & \dots & AB_{2,m} \\ \vdots & \vdots & \dots & \vdots \\ AB_{m,1} & AB_{m,2} & \dots & AB_{m,m} \end{pmatrix}.$$

Proof. It suffices to consider $n = g+1$. Let $\{e_1, \dots, e_g\}$ denote the standard basis for $\mathbb{R}^g$ and let

$$X_j = \begin{pmatrix} 0 & e_j^{\mathrm{T}} \\ e_j & 0 \end{pmatrix}.$$

Here the 0 in the lower left hand corner is the $g \times g$ zero so that X_j is a $(g+1) \times (g+1)$ matrix. We have

$$L_\Omega(X) = \begin{pmatrix} 0 & \Omega_1 & \Omega_2 & \dots & \Omega_g \\ \Omega_1 & 0 & 0 & \dots & 0 \\ \Omega_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \Omega_g & 0 & 0 & \dots & 0 \end{pmatrix}$$

and similarly,

$$L_\Lambda(X) = \begin{pmatrix} 0 & \Lambda_1 & \Lambda_2 & \dots & \Lambda_g \\ \Lambda_1 & 0 & 0 & \dots & 0 \\ \Lambda_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \Lambda_g & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Note that $L_\Omega(X)$ is a $(g+1) \times (g+1)$ block matrix with $d \times d$ matrix entries and is thus $(g+1)d \times (g+1)d$ matrix; whereas $L_\Lambda(X)$ is a $(g+1) \times (g+1)$ block matrix with $k \times d$ matrix entries and is thus a $(g+1)k \times (g+1)d$ matrix.

By the previous Lemma, the condition $L_\Omega(X)h = 0$ implies $L_\Lambda(X)h = 0$, implies that there is a $(g+1) \times (g+1)$ matrix M with $k \times d$ matrix entries $M_{j,m}$, $j, m = 0, 1, 2, \dots, g$ so that $L_\Lambda(X) = ML_\Omega(X)$. By equating the $(0, m)$ entries $m \ge 1$ (that is the entries along the first row) we see that

$$\Lambda_j = M_{00}\Omega_j.$$

□

11. Singularities of the Butterfly Realization

The main result of this and the next section finishes proving the singularities conclusion of our main theorem, Theorem 3.3.

The set up is as follows. Let $\Lambda_1, \dots, \Lambda_g$ denote $1 \times d$ matrices with entries from $\mathbb{R}$; naturally $\Lambda_j^{\mathrm{T}} \in \mathbb{R}^d$. Let L_Λ denote the corresponding pencil,

$$L_\Lambda(x) = \sum_1^g \Lambda_j x_j.$$

The main result of these two sections is.

Proposition 11.1. *If*

$$(x) = L_\Lambda(x)(I - L_A(x))^{-1}L_\Lambda(x)^{\mathrm{T}} \tag{11.1}$$

is a minimal pure butterfly realization of the symmetric rational function $\mathfrak{r}$*, then* $\mathcal{P}$ *equals* $\mathcal{F}^0_{\mathfrak{r}}$*. Recall that* $\mathcal{P}$ *is* $\{X \in (\mathbb{SR}^{n\times n})^g : I - L_A(X) \succ 0\}$*.*

Proof of the singularities conclusion of Theorem 3.3 The descriptor part of the singularities statement was proved earlier in Proposition 9.1. As for the singularity statement for a pure butterfly realization, Theorem 3.3 implies that $\mathfrak{r}$ has a pure minimal butterfly realization. Since we are only concerned with singularities we need only consider the last term of this realization which has the form in (11.1). Thus Proposition 11.1 yields the singularity result. □

11.1. **Rephrasing of Proposition 11.1.** Recall the definition of $\mathcal{P}$ from Theorem 3.3. The boundary of the strict positivity set $\mathcal{P}$ of a linear pencil $I_d - L_A(x)$ is

$$\partial\mathcal{P}^n := \{X \in (\mathbb{SR}^{n\times n})^g : I - L_A(X) \succeq 0,\ \ker(I - L_A(X)) \neq (0)\} \tag{11.2}$$

and

$$\partial\mathcal{P} := \bigcup_{n\geq 0} \partial\mathcal{P}^n.$$

The form of Proposition 11.1 that we actually prove is the following.

Proposition 11.2. *If*

$$(x) = L_\Lambda(x)(I - L_A(x))^{-1}L_\Lambda(x)^{\mathrm{T}} \tag{11.3}$$

is a minimal pure butterfly realization of the symmetric rational function $\mathfrak{r}$ *and* r *is a rational expression for* $\mathfrak{r}$*, then* $\mathcal{F}_{r,for}$ *and* $\partial\mathcal{P}$ *are disjoint; i.e., there does not exists an* X *in both the formal domain of* r *and* $\partial\mathcal{P}$*.*

11.1.1. *Proposition 11.2 implies Proposition 11.1.* Note that $\mathcal{P}$ is contained in $\mathcal{F}^0_{\mathfrak{r}}$ simply because $\mathcal{P}$ is the formal domain of the butterfly realization . Thus, to show that Proposition 11.2 implies Proposition 11.1 it suffices to show if there is a Y in $\mathcal{F}^0_{\mathfrak{r}}$ which is not in $\mathcal{P}$, then there is an X and a rational expression r for $\mathfrak{r}$ such that X is in both $\mathcal{F}_{r,\mathrm{for}}$ and $\partial\mathcal{P}$.

Accordingly, suppose $Y \in \mathcal{F}^0_{\mathfrak{r}} \setminus \mathcal{P}$. In particular, $I - L_A(Y)$ is not positive definite. This means there is a rational expression s equivalent to such that Y is in the formal domain of s. Since $\mathcal{F}^0_{\mathfrak{r}}$ is open and connected and contains 0, there is a $0 < t \leq 1$ for which tY is in both $\mathcal{F}^0_{\mathfrak{r}}$ and $\partial\mathcal{P}$, that is, $I - L_A(tY)$ is positive semidefinite and has a nontrivial kernel. Let $X = tY$. Since X is in the domain of $\mathfrak{r}$, there is a rational expression r which is equivalent to and such that X is in the formal domain of r. (Note, possibly X while in $\mathcal{F}_{r,\mathrm{for}}$ is not in $\mathcal{F}^0_{r,\mathrm{for}}$, however, this does not effect our proof.)

11.1.2. *Buried Singularities Redux.* Given a rational expression r for the rational function $\mathfrak{r}$ of Proposition 11.2, let $\mathcal{B}_r$ denote the collection of all g tuples X in both $\mathcal{F}_{r,\text{for}}$ and $\partial\mathcal{P}$. We call $\mathcal{B}_r$ **the buried singularity set of the butterfly realization relative to** r ; often we abbreviate this to buried singularity set.

The notation $\mathcal{B}_r$ only references the rational expression r. However, r determines the equivalence class $\mathfrak{r}$ and, by Proposition 4.5 the definition of $\mathcal{B}_r$ does not depend upon the choice of minimal pure butterfly realization with $J = I$ for $\mathfrak{r}$. Note also, this definition of buried set is the same as in §9 except here there is an additional non-negativity condition. Proposition 11.2 asserts that $\mathcal{B}_r$ is empty.

11.2. **Definitions and Outline of the Proof of Proposition 11.2.** Fix a positive integers g and d and let $A_1, \dots, A_g$ be given $d \times d$ symmetric matrices with real entries.

Definition 11.3. The tuple A is **irreducible** if, for each nonzero vector $h \in \mathbb{R}^d$,
$$\operatorname{span}(\{A^\alpha h : \alpha\}) = \mathbb{R}^d.$$
Note that irreducible implies unpinned but not conversely.

We shall heavily use a decomposition of vectors v in $\mathbb{R}^{nd}$ as

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} \in \mathbb{R}^{nd} = \oplus_1^d \mathbb{R}^n. \tag{11.4}$$

The proof of Proposition 11.1 breaks into two parts. The first part, the subject of this section, is the following proposition. Assuming $(\mathfrak{r},\ ,r)$ satisfies the hypothesis of Proposition 11.1, but $\mathcal{B}_r$ is not empty, it does two things. One is it replaces the triple $(\mathfrak{r},\ ,r)$ with a triple $(\check{\mathfrak{r}},\check{\ },\check{r})$ which satisfies the proposition and for which $\mathcal{B}_{\check{r}}$ is not empty and so that $\check{\ }$ has irreducible pieces containing the singularity structure of . The second is construction of the "coefficients" of the descriptor realization of a new rational function $\mathfrak{q}$ which has a particular singularity structure.

Proposition 11.4. *Suppose $\mathfrak{r}$ is a rational function with minimal pure butterfly realization as in the hypothesis of Proposition 11.1. If the conclusion of Proposition 11.2 fails for the pair $(\mathfrak{r},\ )$, that is, if there exists a rational expression r for $\mathfrak{r}$ such that $\mathcal{B}_r$ is not empty, then there exists*

(rf) *a rational function $\check{\mathfrak{r}}$;*
(br) *an unpinned pure butterfly realization*
$$\check{\ }(x) = L_{\check{\Lambda}}(x)(I - L_{\check{A}}(x))^{-1} L_{\check{\Lambda}} \mathrm{T}(x)$$
for $\check{\mathfrak{r}}$ acting on $\mathbb{R}^{\check{d}}$;
(dec) *a decomposition of $\check{A} \in \mathcal{S}_g^{\check{d}}$ as a direct sum of irreducible components which we denote $A^k \in \mathcal{S}_g^{n_k}$ for $k = 1, 2, \dots, N$;*

(rex) *a rational expression* $\check{r}$; *and*
(vec) *a nonzero vector* $\lambda \in \mathbb{R}^{\check{d}}$

with the following properties.

(1) *The set* $\mathcal{B}_{\check{r}}$ *is not empty;*
(2) *if* $Y \in \mathcal{B}_{\check{r}}$, *then* $I - L_{A^k}(Y)$ *is not invertible for each* k*;*
(3) *if* $Y \in \mathcal{B}_{\check{r}}^n$ *and* $L_{\check{A}}(Y)v = v$, *then*

$$0 = \langle \lambda, v\rangle = \sum \lambda_j v_j;$$

Here the $v = \sum e_j \otimes v_j = \oplus_1^d v_j$ *is the decomposition in (11.4), that is, for* $Y \in \mathcal{B}_{\check{r}}^n$ *each* v_j *is in* $\mathbb{R}^n$.

The proof of Proposition 11.4 concludes in §11.4.

The full proof of Proposition 11.1 concludes in §12.4 but now we give the rough idea. Begin with $\check{\mathfrak{r}}$ and $\check{r}$ as in the conclusion of Proposition 11.4. Write $\lambda = \oplus\lambda^k$ with respect to the decomposition of $\check{A}$ as A^k. In particular not all λ^k are zero, consequently the rational function determined by the descriptor realization,

$$(x) = \lambda^{\mathrm{T}}(I - L_{\check{A}}(x))^{-1}\lambda = \sum (\lambda^k)^{\mathrm{T}}(I - L_{A^k}(x))^{-1}\lambda^k = \sum {}^k(x)$$

is not zero. In §12 below it is shown, assuming $\mathcal{B}_{\check{r}}$ is not empty, that there are sufficiently many $X \in \mathcal{B}_{\check{r}}$ for which

$$\lim_{t\to 1^-} {}''(tX)[H]$$

exists to invoke the proof of Proposition 9.1 and conclude that $\mathcal{B}_{\check{r}}$, and therefore $\mathcal{B}_r$, must be empty.

11.3. **Phantom Poles of the Butterfly Realization.** A pair (X, v) with $X \in \mathcal{F}_{\mathfrak{r}}^0$ and v nonzero satisfying $L_A(X)v = v$ is a phantom pole of $\mathfrak{r}$ and we wish to show there are none. (Here, X and v have all real entries and X is a tuple of symmetric matrices.) Of course, (X, v) is a phantom pole of $\mathfrak{r}$ if and only if there is a rational expression r for $\mathfrak{r}$ so that $X \in \mathcal{B}_r$ the buried singularity set for r.

We start our proof by recalling the pure butterfly realization

$$(x) = L_\Lambda(x)(I - L_A(x))^{-1}L_\Lambda(x)^{\mathrm{T}};$$

for $\mathfrak{r}$; however, from here until §11.4 we do **not** assume that the representation is minimal.

11.3.1. *Linear Combinations of Null Vectors.*

Lemma 11.5. *If* $X \in \mathcal{F}_{\mathfrak{r}}^0$ *and* $L_A(X)v = v$, *then* $L_\Lambda(X)v = 0$. *Further, if* $H \in \mathbb{S}^g$ *satisfies* $L_A(H)v = 0$, *then* $L_\Lambda(H)v = 0$.

Proof. Fix X and without loss of generality assume $v \neq 0$. There is a rational expression r for $\mathfrak{r}$ such that X is in the formal domain of r. Let $\mathcal{E}$ denote $\{v : L_A(X)v = v\}$. Our hypothesis is that $\mathcal{E}$ is nontrivial. Decompose relative to $\mathcal{E} \oplus \mathcal{E}^\perp$ and use the fact that for t near 1, but $t \neq 1$, that $I - tL_A(X)$ is invertible to write,

$$(I - tL_A(X))^{-1} = \begin{pmatrix} \frac{1}{1-t} & 0 \\ 0 & (I - tQ)^{-1} \end{pmatrix}$$

for some Q for which $I - Q$ is invertible. Using $L_\Lambda(tX) = tL_\Lambda(X)$, it now follows that

$$(tX) = \frac{t^2}{1-t} L_\Lambda(X) P_{\mathcal{E}} L_\Lambda(X)^{\mathrm{T}}$$

is bounded near $t = 1$ and hence $0 = L_\Lambda(X) P_{\mathcal{E}} L_\Lambda(X)^{\mathrm{T}}$. Thus, $L_\Lambda(X)v = 0$ for each $v \in \mathcal{E}$.

Next suppose $L_A(H)v = 0$ too. For small enough $s \in \mathbb{R}$ we have $X + sH \in \mathcal{F}_r^0$ and moreover $L_A(X+sH)v = v$. Thus $L_\Lambda(X+sH)v = 0$ and we conclude $L_\Lambda(H)v = 0$. □

Lemma 11.6. *Suppose r is a rational expression for $\mathfrak{r}$ and $X^1, \dots, X^m \in \mathcal{B}_r$ all act on $\mathbb{R}^n$ and $L_A(X^j)v_j = v_j$ for $j = 1, \dots m$. If $H \in \mathbb{S}^g$ satisfies $L_A(H)(\sum c_j v_j) = 0$, then $L_\Lambda(H)(\sum c_j v_j) = 0$.*

Proof. Let Z denote the g-tuple of $(m+1)\times(m+1)$ block diagonal matrices Z_i with $n \times n$ matrix entries $0, X_i^1, X_i^2, \dots, X_i^m$ down the main diagonal. Assume $L_A(H)(\sum_k c_k v_k) = 0$. Let

$$\eta = \begin{pmatrix} 0 \\ v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}.$$

Note that $L_A(Z)\eta = \eta$. Let

$$\begin{aligned} Y_j &= \begin{pmatrix} 0 & c^{\mathrm{T}} H_j \\ cH_j & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & c_1 H_j & c_2 H_j & \dots & c_m H_j \\ c_1 H_j & 0 & 0 & \dots & 0 \\ c_2 H_j & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_m H_j & 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{for } \ j = 1, \dots, g. \end{aligned}$$

Here

$$c^{\mathrm{T}} = \begin{pmatrix} c_1 & c_2 & \dots & c_m \end{pmatrix}$$

and so Y_j is an $(m+1) \times (m+1)$ block matrix with $n \times n$ matrix entries (remember H, and thus Y, is a g-tuple of matrices). We have,

$$L_A(Y) = \begin{pmatrix} 0 & c_1 L_A(H) & \dots & c_m L_A(H) \\ c_1 L_A(H) & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ c_m L_A(H) & 0 & \dots & 0 \end{pmatrix}.$$

Hence,

$$L_A(Y)\eta = \begin{pmatrix} \sum_k c_k L_A(H) v_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, as $L_A(H)(\sum c_k v_k) = 0$, direct computation gives $L_A(Y)\eta = 0$. Thus Lemma 11.5 forces $L_\Lambda(Y)\eta = 0$. But $L_\Lambda(Y)\eta = 0$ by a (similar) direct computation is equivalent to $L_\Lambda(H)(\sum_k c_k v_k) = 0$. ☐

11.3.2. *A Universal Dependence Relation.* The decomposition of vectors v in $\mathbb{R}^{nd}$ as $v = \oplus_1^d v_j$ with each $v_j \in \mathbb{R}^n$ has the following strong property.

Proposition 11.7. *Let r denote a rational expression for $\mathfrak{r}$. If the buried singularity set $\mathcal{B}_r$ is not empty, then there exists a $\lambda \in \mathbb{R}^d$, $\lambda \neq 0$ such that for any $Y \in \mathcal{B}_r$ and u satisfying $L_A(Y)u = u$ we have*

$$\sum_{j=1}^d \lambda_j u_j = 0$$

We start with a lemma which requires a definition.

Definition 11.8. Given $v \in \mathbb{R}^n \otimes \mathbb{R}^d$, write, $v = \oplus_1^d v_j$ with each $v_j \in \mathbb{R}^n$. Then for $H \in \mathbb{R}^{n\times n}$

$$(H \otimes I_d)v = \begin{pmatrix} Hv_1 \\ Hv_2 \\ \vdots \\ Hv_d \end{pmatrix}.$$

We say a subset $\mathcal{S} \subset \mathbb{R}^n \otimes \mathbb{R}^d$ is a **left operator module** provided $(H \otimes I_d)\mathcal{S} \subset \mathcal{S}$ for every $H \in \mathbb{R}^{n\times n}$. As a remark, note that it is enough that $(H \otimes I_d)\mathcal{S} \subset \mathcal{S}$ for a set of H which spans $\mathbb{R}^{n\times n}$.

The result required about left operator modules is the following.

Lemma 11.9. *If $\mathcal{V} \subset \mathbb{R}^n \otimes \mathbb{R}^d$ is a invariant under $\mathbb{R}^{n\times n}$; that is, $(H \otimes I_d)\mathcal{V} \subset \mathcal{V}$ and if $\mathcal{V}$ is a proper subset of $\mathbb{R}^n \otimes \mathbb{R}^d$, then there is a vector $C \in \mathbb{R}^d$ for which $([1] \otimes C)$ is orthogonal to $\mathcal{V}$; i.e., for each $v \in \mathcal{V}$,*

$$0 = \sum_{j=1}^d C_j v_j.$$

Proof. There exists φ orthogonal to $\mathcal{V}$. Write

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_d \end{pmatrix}.$$

Given $H \in \mathbb{R}^{n\times n}$ and $v \in \mathcal{V}$, we have

$$\langle (H \otimes I)\varphi, v\rangle_{\mathbb{R}^{nd}} = \langle \varphi, (H^{\mathrm{T}} \otimes I)v\rangle_{\mathbb{R}^{nd}} = 0.$$

Choose $H = \psi\varphi_1^{\mathrm{T}}$ to obtain

$$0 = \langle(\psi\varphi_1^{\mathrm{T}} \otimes I_n)\varphi, \rangle_{\mathbb{R}^{nd}} = \sum_j^d \langle \psi\varphi_1^{\mathrm{T}}\varphi_j, v_j\rangle_{\mathbb{R}^{nd}} = \sum \psi(\varphi_1^{\mathrm{T}}\varphi_j)v_j,$$

when we have used that $\varphi_1^{\mathrm{T}}\varphi_j$ is a scalar.

Since this is true for every ψ, we conclude

$$0 = \sum_j (\varphi_1^{\mathrm{T}}\varphi_j)v_j$$

So choosing

$$C = \begin{pmatrix} \varphi_1^{\mathrm{T}}\varphi_1 \\ \varphi_1^{\mathrm{T}}\varphi_2 \\ \vdots \\ \varphi_1^{\mathrm{T}}\varphi_d \end{pmatrix}$$

completes the proof. $\square$

Lemma 11.10. *Fix n and let $\mathcal{V}_n$ denote the set*

$$\mathcal{V}_n = \{v \in \mathbb{R}^{nd} : L_A(X)v = v \text{ for some } X \in \mathcal{B}_r\}.$$

Either $\mathcal{V}_n$ spans $\mathbb{R}^{nd}$ or for each v satisfying $L_A(X)v = v$ the set $\{v_1, \dots, v_d\}$ is linearly dependent.

Proof. Let n denote the dimension of the space that X acts on. In particular $X = (X_1, \dots, X_g)$ is a g-tuple of symmetric $n \times n$ matrices.

We observe that if $v \in \mathcal{V}_n$ and U is a $n\times n$ unitary matrix, then $(U^{\mathrm{T}} \otimes I_d)v \in \mathcal{V}_n$, since $U^{\mathrm{T}}XU \in \mathcal{B}_r$ and $L_A(U^{\mathrm{T}}XU)(U^{\mathrm{T}} \otimes I_d)v = (U^{\mathrm{T}} \otimes I_d)v$. It follows that if $\lambda \in \mathcal{V}_n^{\perp}$, then $(U \otimes I_d)\lambda \in \mathcal{V}_n^{\perp}$. Since any matrix can be written as a linear combination of unitary matrices, $\mathcal{V}_n^{\perp}$ is a left operator module. By Lemma 11.9, if $\mathcal{V}_n$ does not span $\mathbb{R}^{nd}$, then there is a $C \in \mathbb{R}^d$ such that $Cv = \sum_j c_j v_j = 0$ for all $v \in \mathcal{V}_n$. $\square$

We use the preceding two lemmas to prove another lemma.

Lemma 11.11. *Let r denote a rational expression for $\mathfrak{r}$ and suppose $\mathcal{B}_r$ is not empty. If $X \in \mathcal{B}_r$ and $v \in \mathbb{R}^{nd}$ satisfy $L_A(X)v = v$, then $\{v_1, \dots, v_d\}$ is linearly dependent.*

Proof. We argue by contradiction. Accordingly suppose $X \in \mathcal{B}_r$, $L_A(X)v = v$ and $\{v_1, \dots, v_d\}$ are linearly independent. By replacing X by $\oplus_1^m X$ and v by $\oplus_1^m v$, and n by mn, where $mn \geq g$, we may assume that $n > g$.

Independence of $\{v_1, \dots, v_d\}$ implies, by Lemma 11.10, that $\mathcal{V}_n$, as defined in the Lemma, spans $\mathbb{R}^{nd}$ (of course it need not be a subspace). Since $\mathcal{V}_n$ spans $\mathbb{R}^{nd}$, Lemma 11.6 implies that if v in $\mathbb{R}^{nd}$ and H in $(\mathbb{SR}^{n\times n})^g$ satisfy $L_A(H)v = 0$, then $L_\Lambda(H)v = 0$. Hence, as $n > g$, by the Nullpencilsatz (Proposition 10.1) there is a C such that $L_\Lambda = CL_A$.

For each $v \in \mathcal{V}_n$ there is an X satisfying $L_A(X)v = v$. By Lemma 11.5, $L_\Lambda(X)v = 0$. Using both $L_A(X)v = v$ and $L_\Lambda(X)v = 0$ we find

$$0 = L_\Lambda(X)v = CL_A(X)v = (C \otimes I)v.$$

Since $\mathcal{V}_n$ spans $\mathbb{R}^{nd}$, it follows that $C = 0$ and therefore $L_\Lambda = 0$, contradicting the fact that $L_\Lambda \neq 0$ which is true because of minimality. □

Proof of Proposition 11.7 The collection $\mathcal{B}_r$ is closed with respect to direct sums and for each $(X, v) \in \mathcal{V}$, the set $\{v_1, \dots, v_d\}$ is linearly dependent by Lemma 11.11. The proposition now follows from an application of the Theorem 6.6 on linear dependence. □

11.4. **Proof of Proposition 11.4.** Proposition 11.4 concerns a rational function $\mathfrak{r}$ with the pure butterfly realization

$$(x) = L_\Lambda(x)(I - L_A(x))^{-1} L_\Lambda(x)^{\mathrm{T}},$$

where

$$L_A(x) = \sum_1^g A_j x_j \qquad L_\Lambda(x) = \sum_1^g \Lambda_j x_j,$$

for $A = (A_1, \dots, A_g)$ a tuple of $d \times d$ symmetric matrices (not necessarily irreducible) and $\Lambda = (\Lambda_1, \dots, \Lambda_g)$ with $\Lambda_j^{\mathrm{T}} \in \mathbb{R}^d$. Now we do assume that the representation is minimal.

We decompose A into irreducible summands A^k, $k = 1, 2, \dots, N$, such that

$$(x) = \sum_1^N {}^k(x) = \sum_1^N L_{\Lambda^k}(x)(I - L_{A^k}(x))^{-1} L_{\Lambda^k}(x)^{\mathrm{T}}, \tag{11.5}$$

where

$${}^k(x) = L_{\Lambda^k}(x)(I - L_{A^k}(x))^{-1} L_{\Lambda^k}(x)^{\mathrm{T}}.$$

Observe that the minimality of implies each Λ^k is nonzero. (This is all of minimality that we use).

For $J \subset \{1, 2, \dots, N\}$ a nonempty set, define

$$_J := \sum_{k \in J} {}^k \quad \textit{and} \quad \tilde{}_J := \sum_{j \notin J} {}^k = - {}_J.$$

With $J_N = \{1, 2, \dots, N\}$, the hypothesis of Proposition 11.4 says that the rational expression $_{J_M}$ has a nonempty buried singularity set. Let

$M = \min(\{|J| : \exists$ a rational expression s equivalent to $_J$ such that $\mathcal{B}_s \neq \emptyset\})$.

Note that $M \geq 1$ and there is a J satisfying $|J| = M$ and so there is a rational expression s for this J so that $\mathcal{B}_s \neq \emptyset$. Without loss of generality, we may assume that $J = J_M = \{1, 2, \dots, M\}$. Finally we define the butterfly realization

$$\check{} := s_{J_M}$$

and let, as expected, $\check{\mathfrak{r}}$ denote the corresponding rational function and $\check{r}$ a rational expression for $\check{\mathfrak{r}}$ for which $\mathcal{B}_{\check{r}}$.

Proof of item 1: $\mathcal{B}_{\check{r}}$ is nonempty by construction.

Proof of item 2: We shall show that if $X \in \mathcal{B}_{\check{r}}$, then each $I - L_{A^k}(X)$ is not invertible. To verify this, suppose $I - L_{A^M}(X)$ is invertible (and thus is positive definite). Consider the rational expression

$$u = \check{r} - \ _{\{M\}}.$$

Since X is in both domains on the right hand side, it is in the domain of u. Further, u is a rational expression for $\ _{J_{M-1}}$, since, near 0, the expressions agree. (An appeal to formal power series expansions shows this as well.) It follows that X is in the domain of the rational expression u equivalent to $\ _{J_{M-1}}$ while at the same time for some $1 \le k < M$, the matrix $I - L_{A^k}(X)$ is not invertible. Hence $\mathcal{B}_u$ is not empty, contrary to the choice of M.

Prove item 3 by applying Proposition 11.7 to $\check{r}$.

12. Singularities of the Butterfly Realization: An Auxiliary Function

This section begins with the pure butterfly realization

$$\check{\ }(x) = \sum \ ^k(x) = \sum_1^N L_{\Lambda^k}(x)(I - L_{A^k}(x))^{-1} L_{\Lambda^k}(x)^{\mathrm{T}}$$

and the λ produced by Proposition 11.4. Define a rational function $\mathfrak{q}$ by the descriptor realization

$$\ (x) := \lambda^{\mathrm{T}}(I - L_A(x))^{-1}\lambda.$$

Decompose λ compatibly with the A^k and write as a sum of descriptor realizations:

$$\ (x) := \sum_1^k (\lambda^k)^{\mathrm{T}}(I - L_{A^k}(x))^{-1}\lambda^k \tag{12.1}$$

(the lack of precision in this statement is, we hope, forgivable, since both sides have the same domain). Recall also that this $\check{\ }$ has the property that if $\check{r}$ is a rational expression for $\mathfrak{r}$ and if $X \in \mathcal{B}_r$ (where r is a rational expression for $\mathfrak{r}$ for which $\mathcal{B}_r$ is not empty per Proposition 11.4), then each $I - L_{A^k}(X)$ is not invertible. Further, each A^k is irreducible from which it follows that, for each k, either $\lambda^k = 0$ or $\ ^k(x) = (\lambda^k)^{\mathrm{T}}(I - L_{A^k}(x))^{-1}\lambda^k$ is minimal.

The objective of this section is to show that $\mathfrak{q}$ has no singularities at certain X where $I - L_{A^k}(X)$ is singular, thereby contradicting the fact (proved in §9) that descriptor realizations have no buried singularities. This will prove Proposition 11.1.

Henceforth in this section we denote $\check{\mathfrak{r}}$ simply by $\mathfrak{r}$. Thus we now take r to be a function with all of the properties of $\check{\mathfrak{r}}$ and its butterfly realization $\check{\ }$ concluded in Proposition 11.4.

Lemma 12.1. *If r is a rational expression for $\mathfrak{r}$ and if $X \in \mathcal{B}_r$, then $\ (tX)$ is bounded for t near 1.*

Proof. Because $I - L_A(tX) = I - tL_A(X)$ is symmetric, that for t near 1, but $t \neq 1$, $(I - L_A(tX))^{-1}$ is defined. On the other hand, by decomposing $\mathbb{R}^{nd}$ into $\mathcal{K} \oplus \mathcal{K}^{\perp}$, where $\mathcal{K}$ is the kernel of $I - L_A(X)$ and using $\lambda^{\mathrm{T}} v = 0$ for $v \in \mathcal{K}$, it follows that

$$(tX) = \begin{pmatrix} 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} \frac{1}{1-t} P_{\mathcal{K}} & 0 \\ 0 & ([I - P_{\mathcal{K}}] + tG)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ \lambda_1^{\mathrm{T}} \end{pmatrix}$$

where $([I - P_{\mathcal{K}}] + tG)$ is invertible at $t = 1$. Thus (tX) is bounded for t near 1. $\square$

Throughout the remainder of this section we fix a rational expression r for $\mathfrak{r}$. To complete the proof of Proposition 11.1 it suffices to show that $\mathcal{B}_r$, the buried singularity set of $\mathfrak{r}$ with respect to r is empty. Accordingly, to obtain a contradiction, assume that $\mathcal{B}_r$ is not empty.

12.1. **Minimal Kernels and Irreducible Cut Outs.** Define

$$M = \min(\{\dim(\ker(I - L_A(X))) : X \in \mathcal{B}_r\}).$$

Here the minimum is taken over all dimensions n with $X \in (\mathbb{SR}^{n \times n})^g \cap \mathcal{B}_r$. Let

$$\mathcal{B}_r(M) = \{X \in \mathcal{B}_r : M = \dim(\ker(I - L_A(X)))\}.$$

Define

$$D(X) := \det(I - L_A(X))$$

thought of as a mapping $D : (\mathbb{SR}^{n \times n})^g \to \mathbb{R}$. To be precise there is one such function for each n but to conserve notation (and sanity) we shall denote them all by the same symbol D. Indeed, that the action takes place in a specific $(\mathbb{SR}^{n \times n})^g$ will often not be explicit from the notation.

In the Lemma below and in what follows $Z(F)$ denotes the zero set of a rational function $F : \mathbb{R}^N \to \mathbb{R}$.

Lemma 12.2. *If $X \in \mathcal{B}_r(M)$, then there is an irreducible polynomial F which divides D and an open set $U \ni X$ so that if $Y \in U \cap \mathcal{B}_r$, then $F(Y) = 0$, but $F(tY)$ vanishes to order one at $t = 1$ and so the directional derivative of F at Y in the direction Y, $F'(Y)[Y]$, is not zero.*

Moreover, the open set $U \ni X$ can be chosen to satisfy

$$U \cap Z(F) = U \cap \partial \mathcal{P} = U \cap \mathcal{B}_r.$$

Finally, F^M divides D as

$$D = F^M Q$$

where Q does not vanish in a neighborhood of X.

Proof. Choose a neighborhood W of X which lies entirely in the domain of $\mathfrak{r}$. There is a $1 > \delta > 0$ and a neighborhood V of X contained in W so that if $Y \in V$ and $|s| < \delta$, then $(1 + s)Y \in W$. Since $I - L_A(X) \succeq 0$ and its kernel has dimension M, there is a neighborhood U of X contained in V such that if $Y \in U$, then $I - L_A(Y)$ has exactly M eigenvalues less than δ. If $I - L_A(Y)$ is not positive semidefinite, then there is an $0 < s < \delta < 1$ so that

$I - L_A((1-s)Y) \succeq 0$ and has a kernel. From the construction, $(1-s)Y \in \mathcal{B}_r$ and therefore $(1-s)Y \in \mathcal{B}_r(M)$.

Without loss of generality, the upper $(N-M)\times(N-M)$ submatrix of $I - L_A(X)$ is invertible. For future reference note that as $I - L_A(X) \succeq 0$, this upper $(N-M)\times(N-M)$ submatrix must be positive definite. By shrinking the neighborhood U if necessary, we can assume that the upper $(N-M)\times(N-M)$ submatrix of $I - L_A(Y)$ is invertible for all $Y \in U$. Let P denote the projection onto the first $N-M+1$ coordinates and let

$$G(Y) := \det(P(I - L_A(Y))P).$$

This forces $G(Y) = 0$ on $Y \in U \cap \partial\mathcal{P}$, since $P(I - L_A(Y))P$ has a kernel.

Next suppose $Y \in U$ and $G(Y) = 0$. If $I - L_A(Y) \succeq 0$, then $Y \in \partial\mathcal{P}$. On the other hand, if $I - L_A(Y)$ is not positive semidefinite, then there is an $0 < s < 1$ so that $(1-s)Y \in \mathcal{B}_r(M)$ so that $G((1-s)Y) = 0$. On the other hand,

$$P(I - L_A((1-s)Y))P = (1-s)P(I - L_A(Y))P + sP \succ 0$$

is positive definite so that $G((1-s)Y) > 0$. Thus, for $Y \in U$, $G(Y) = 0$ if and only if $Y \in \mathcal{B}_r$.

Note now that $g(t) = G(tX)$ satisfies $g(1) = 0$, and $g(t) > 0$ for $0 \le t < 1$. If $g(t) > 0$ for $t > 1$, then g has a double zero at 1 which contradicts the choice of M in that

$$P_-(I - L_A(X))P_- \succ 0,$$

where P_- is the projection onto the first $N-M$ coordinates. It follows that

$$\frac{dG(tX)}{dt}_{|t=1} \neq 0. \tag{12.2}$$

Factor $G = f_1 \dots f_\ell$ as a product of irreducible real polynomials and without loss of generality assume that $f_1(X) = 0$. If say $f_2(X) = 0$ also, then (12.2) is violated. Thus by choosing a neighborhood of X even smaller than U if necessary, we can assume that if $Y \in U$, then $G(Y) = 0$ if and only if $f_1(Y) = 0$. For notational ease, let $F = f_1$. Now F is irreducible and, in the neighborhood U, $F(Y) = 0$ implies $D(Y) = 0$. Since also $F' \neq 0$ on $U \cap Z(F)$, we conclude from Theorem 17.1 in Appendix 17 that F divides D.

Since F divides D, there is a D_1 so that $D = FD_1$. Fix $Y \in U \cap \mathcal{B}_r$. The polynomial $p(t) = D(tY)$ has a zero of order M at $t = 1$, whereas $F(tY)$ has a zero of order one at $t = 1$. Hence, $D_1(tY)$ has a zero of order $M-1$. Thus, assuming $M > 1$, if $Y \in U$ and $F(Y) = 0$, then $D_1(Y) = 0$. Hence, F divides D_1. Continuing in this fashion, we find

$$D = F^M Q$$

for some Q which does not vanish at X. $\square$

12.2. **The Main Result on q.** Recall the rational function q determined by the butterfly realization

$$(x) = \lambda^{\mathrm{T}}(I - L_A(x))^{-1}\lambda.$$

For n fixed, let

$$\mathsf{q} : (\mathbb{SR}^{\mathrm{n}\times\mathrm{n}})^{\mathrm{g}} \to \mathbb{SR}^{\mathrm{n}\times\mathrm{n}}$$

denote the evaluation,

$$\mathsf{q}(X) = (I \otimes \lambda^{\mathrm{T}})(I - L_A(X))^{-1}(I \otimes \lambda).$$

Thus, q is a rational function of $\frac{n(n-1)}{2}g$ real variables. Again, precision requires a different symbol q for each n, but for simplicity we denote these all by q with the n understood.

The following Proposition is the main result of this section.

Proposition 12.3. *Fix $\hat{X} \in \mathcal{B}_r(M)$ and a positive integer K and let*

$$\hat{X}^K = I_K \otimes \hat{X}$$

(the $K \times K$ block diagonal matrix with $\hat{X}$ as the diagonal entries).

Then there is a neighborhood $W' \subset (\mathbb{SR}^{Kn\times Kn})^g$ containing $\hat{X}^K$ on which the entries of q are rational functions $\frac{a}{b}$, where b does not vanish on W'. In particular, q is C^2 in a neighborhood of $\hat{X}^K$.

By choosing K large enough to make $I - L_A(\hat{X}^K)$ have sufficiently large kernel we will be able to proceed with our earlier argument for descriptor systems given to prove Proposition 9.1 to finish the proof of Proposition 11.1.

In view of Lemma 12.1 it suffices to prove the following Lemma.

Lemma 12.4. *Fix $\hat{X} \in \mathcal{B}_r(M)$ and a positive integer K and let*

$$\hat{X}^K = I_K \otimes \hat{X}$$

If there is a neighborhood $W \subset (\mathbb{SR}^{Kn\times Kn})^g$ of $\hat{X}^K$ so that for each $Y \in W \cap \mathcal{B}_r$ the rational function (one real variable) $\mathsf{q}(tY)$ is bounded near $t = 1$, then there is a neighborhood W' of $\hat{X}^K$ on which the entries of q are rational functions $\frac{a}{b}$, where b does not vanish on W'. In particular, q is C^2 in a neighborhood of $\hat{X}^K$.

12.3. **Proof of Lemma 12.4.** For $\tau = (t_1, \ldots, t_K)$, let $\tau\hat{X}$ denote the block diagonal matrix with (j, j) entry $t_j\hat{X}$. In particular, with $[1] = (1, 1, \ldots, 1)$, we write $[1]\hat{X} = \hat{X}^K$. The remainder of the proof is divided into two subsubsections. For fixed dimension, using Cramer's rule $\mathsf{q}(\mathtt{X})$ can be expressed as a matrix of ordinary rational functions (of many variables) with the common denominator $D(X) = \det(I - L_A(X))$. In Subsection 12.3.1 this determinant is analyzed. In the Subsubsection 12.3.2 we show that all the poles of D (near our point of interest) are cancelled by zeros in the denominator.

12.3.1. *The Denominator of* q. Choose a neighborhood $\hat{W}$ of $\hat{X}^K$ such that

$$\hat{W} \cap \mathcal{B}_r = \hat{W} \cap \partial\mathcal{P}.$$

Fix a point $\sigma = (1, t_2, \dots, t_K)$, where each $t_j < 1$ but are close enough to 1 to make $\sigma\hat{X} \in \hat{W}$. Note that $\sigma\hat{X} \in \mathcal{B}_r(M)$ and hence there is a neighborhood U_1 of $\sigma\hat{X}$ and an irreducible monic function F_1 satisfying the conclusion of Lemma 12.2. In particular, F_1^M divides D and $F_1(tY)$ vanishes to order exactly one for $Y \in U_1 \cap Z(F_1)$.

Let $G_1(\tau) = F_1(\tau\hat{X})$. For $\tau = (1, s_2, \dots, s_K)$ near τ_1, we have $\tau\hat{X}$ is in $U \cap \partial\mathcal{P}$ and thus $G_1(\tau) = F_1(\tau\hat{X}) = 0$. It follows that $(1 - s_1)$ divides $G_1(\tau)$. In particular, $F_1(\hat{X}^K) = G_1([1]) = 0$ and $(1 - s_1)^M$ divides $D(\tau\hat{X})$.

Continue this process to obtain for each $j = 1, 2, \dots, K$ when we fix $\tau_j = \{t_1, \dots, t_{j-1}, 1, t_{j+1}, \dots, t_K\}$ there is a similar F_j and open set U_j so that F_j^M divides D. There is the possibility that some of these F_j are the same (up to a nonzero constant multiple). Accordingly, let

$$J_\mu = \{j : F_j = \kappa F_\mu \text{ for some nonzero constant } \kappa\}.$$

Note that

$$F_\mu(t\hat{X}^K) = (1 - t)^{M|J_\mu|},$$

where $|J_\mu|$ is the cardinality of J_μ.

Without loss of generality, we can assume there is an ℓ such that $J_1, \dots, J_\ell$ are distinct (pairwise disjoint) and such that $\cup_1^\ell J_\mu = \{1, \dots, K\}$. For notation ease, let $c_j = |J_j|$, the cardinality of J_j. In particular, $K = \sum_1^\ell c_j$.

We have

$$D = F_1^{c_1 M} F_2^{c_2 M} \dots F_\ell^{c_\ell M} F_0$$

for some polynomial F_0. Further, $(1 - s_j)^M$ divides $D(\tau\hat{X})$ so that

$$D(\tau\hat{X}) = (1 - s_1)^M \cdots (1 - s_K)^M R(\tau),$$

for some polynomial R. Choosing $\tau = [1]$, it follows that

$$D(t[1]\hat{X}) = D(t\hat{X}^K) = (1 - t)^{MK} R(t[1]).$$

Since $D(t\hat{X}^K)$ has a zero of order MK at $t = 1$, it follows that $R(1) \neq 0$. Thus, since $F_j(t\hat{X}^K)$ has a zero at $t = 1$, it follows that $F_0(\hat{X}^K) \neq 0$.

12.3.2. *Zero-Pole Cancellation.* We now return to

$$\begin{aligned} \mathsf{q}(X) = {} & (I \otimes \lambda^{\mathrm{T}})(I - L_A(X))^{-1}(I \otimes \lambda) \\ = {} & \left(\tfrac{p_{\nu,\mu}}{F_1^{c_1 M} F_2^{c_2 M} \dots F_\ell^{c_\ell M} F_0(X)}\right)_{\mu,\nu} \end{aligned} \tag{12.3}$$

and we analyze what happens near the point $\hat{X}^K$. Our hypotheses say that in a neighborhood of this point W and for $Y \in W \cap B_r$, that

$$\lim_{t \to 1} \mathsf{q}(tY)$$

is bounded from which it follows that if $F_j(Y) = 0$, then $p_{\mu,\nu}(Y) = 0$ (all μ, ν). We conclude that each $p_{\mu,\nu}$ vanishes on a $Z(F_j)$ open set (namely U_j) and hence F_j divides $p_{\mu,\nu}$ (each j). Indeed, for $Y \in U_j$, the function $F_j(tY)$

vanishes to order exactly one at $t = 1$. Thus, $p_{\mu,\nu}(tY)$ vanishes to order at least $c_j M$ at $t = 1$.

In particular, $p_{\mu,\nu}$ vanishes on U_j and so F_j divides $p_{\mu,\nu}$:

$$p_{\mu,\nu} = F_j p^j_{\mu,\nu}$$

But now, since as a function of t, $p(tY)$ vanishes to order at least $c_j M$ at $Y \in U_j$, the polynomial $p^j_{\mu,\nu}$ vanishes on the $Z(F_j)$ open set U_j and thus F_j divides $p^j_{\mu,\nu}$. Continuing in this fashion, we conclude that $F_j^{c_j M}$ divides $p_{\mu,\nu}$. It now follows that

$$\mathsf{q}(X) = (\frac{\breve{p}_{\nu,\mu}}{F_0(X)})_{\mu,\nu},$$

where of course $F_0(X)$ doesn't vanish near $\hat{X}^K$. In particular, $\mathsf{q}(X)$ has a continuous second derivative in a neighborhood of $\hat{X}^K$. This proves Lemma 12.4. $\square$

12.4. **Finish the Proof of Proposition 11.1.** Without loss of generality, we may assume that $\lambda^1 \neq 0$. Note that if $X \in \mathcal{B}_r(M)$, then $I - L_{A^1}(X)$ is not invertible (in fact each of $I - L_{A^K}(X)$ is not invertible). Also, recall that the descriptor representation,

$${}^1(x) = (\lambda^1)^{\mathrm{T}}(I - L_{A^1}(x))^{-1}\lambda^1$$

is minimal (and unpinned).

For $X \in (\mathbb{SR}^{n\times n})^g$ for which $I - L_{A^1}(X)$ is invertible we of course define,

$$\Gamma^1(X) = (I - L_{A^1}(X))^{-1}\lambda.$$

We now argue as in Subsection 9.3 using Lemma 9.3. Suppose there is a μ so that for every X with $X_1^2 + \cdots + X_g^2 \prec \varepsilon I$, we have

$$0 = \sum \mu_j \Gamma^1(X)_j.$$

Just as before this violates minimality of the realization for ^{1}. Consequently, there exists an X and v so that $X_1^2 + \cdots + X_g^2 \prec \varepsilon I$

$$w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix} = \Gamma^1(X)v \in \mathbb{R}^d \otimes \mathbb{R}^n$$

has linearly independent entries, that is, $\{w_1, \dots, w_d\}$ is a linearly independent set in $\mathbb{R}^n$.

Now choose $Y \in B_r(M)$, let $K > \frac{d(d-1)}{2}$, and let

$$\hat{X} = \begin{pmatrix} Y & 0 \\ 0 & X \end{pmatrix}.$$

Observe that $\hat{X} \in \mathcal{B}_r(M)$, so Lemma 12.4 applies to the K-th power $\hat{X}^K$ of $\hat{X}$. Therefore for each H, the limit

$$\frac{1}{2}\lim_{t\to 1} {}''(t\hat{X}^K)[H]$$

exists. On the other hand, $\frac{1}{2}\ \ ''(t\hat{X}^K)[H] = \frac{1}{2}\sum(\ ^j)''(t\hat{X}^K)[H]$ and, for $0 < t < 1$ each summand $\frac{1}{2}(\ ^j)''(t\hat{X}^K)[H]$ is positive semidefinite, since $X \in \partial\mathcal{P} \subset$ *closure* $\mathcal{P}$. Thus, for each H fixed

$$(\ ^1)''(t\hat{X}^K)[H]$$

is bounded for $0 < t < 1$. The choice of K implies that the dimension of the kernel of $I - L_{A^1}(\hat{X}^K)$ is at least $K > \frac{d(d-1)}{2}$. Further, there is a v^K such that

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_d \end{pmatrix} = \Gamma^1(\hat{X}^K)v^K$$

has linearly independent entries, since the same is true for a summand. We can now apply the argument behind Lemma 9.3 to obtain a contradiction.

13. Asymptotic Growth of the Butterfly Realization

This section gives proofs for the conclusions in Theorem 3.3 pertaining to growth at infinity of r.

13.1. **Proof of Growth Conclusions in Theorem 3.3.** Let $\mathfrak{r}$ denote a NC scalar rational function which has a butterfly realization with $J = I$. We set about to consider the asymptotics of $\frac{\mathfrak{r}(tX)}{t^2}$ as $t \to \infty$. From Lemma 8.2, $\mathfrak{r}$ has a butterfly realization where L_Λ is linear (has no constant term). Recall that, since $\mathfrak{r}$ is rational, for each tuple X there is a T such that if $t > T$, then $\mathfrak{r}(tX) = \ (tX)$ is defined (since $I - L_A(tX) = I - tL_A(X)$ is linear in t and is invertible at 0).

Now suppose X is in $(\mathbb{SR}^{n\times n})^g$, η is a nonzero vector in $\mathbb{R}^{nd}$, and $L_A(X)\eta = 0$. Decomposing $\mathbb{R}^{nd}$ with respect to the kernel of $L_A(X)$ and its orthocomplement, we have

$$L_A(tX) = tL_A(X) = \begin{pmatrix} 0 & 0 \\ 0 & tP \end{pmatrix}$$

for some invertible matrix P which of course depends on X. Hence, for t large,

$$(I - tL_A(X))^{-1} = \begin{pmatrix} I & 0 \\ 0 & (I - tP)^{-1} \end{pmatrix}.$$

Since L_Λ is linear,

$$L_\Lambda(tX) = tL_\Lambda(X) = t\begin{pmatrix} D & F \end{pmatrix}$$

for some D, F.

$$L_\Lambda(tX)(I - tL_A(X))^{-1}L_\Lambda(tX)^{\mathrm{T}} = t^2DD^{\mathrm{T}} + t^2F(I - tP)^{-1}F^{\mathrm{T}}$$

Substituting this into the butterfly realization for gives,

$$(13.1)\qquad \frac{\ (tX)}{t^2} = \frac{\ _0}{t^2} + \frac{\ _1(tX)}{t^2} + \frac{\ell(tX)\ell(tX)^2}{t^2} + DD^{\mathrm{T}} + F(I - tP)^{-1}F^{\mathrm{T}}.$$

The first two terms and the last term tend to 0 as t tends to infinity. Thus

$$\lim_{t\to\infty} \frac{(tX)}{t^2} = \ell(X)\ell(X)^{\mathrm{T}} + DD^{\mathrm{T}}.$$

Thus we see that $\mathfrak{r}$ has at most second order growth at infinity.

If $\mathfrak{r}$ has at most first order growth, then does also, consequently $D = 0$ and $\ell = 0$. We conclude, $L_\Lambda(X) = 0$ on the kernel of $L_A(X)$; that is, if $L_A(X)v = 0$, then $L_\Lambda(X)v = 0$. By the Nullpencilsatz, Proposition 10.1, there is vector $C^{\mathrm{T}} \in \mathbb{R}^d$ such that $CA_j = \Lambda_j$ for $j = 1, 2, \dots, g$.

Since $L_\Lambda = CL_A$, we can write $L_\Lambda = C - C(I - L_A)$ and obtain,

$$L_\Lambda(x)(I - L_A(x))^{-1}L_\Lambda(x)^{\mathrm{T}} = C(I - L_A(x))^{-1}C^{\mathrm{T}} - 2CC^{\mathrm{T}} + C(I - L_A(x))C^{\mathrm{T}}.$$

Thus,

$$(x) = \tilde{}_0 + \tilde{}_1(x) + C(I - L_A(x))^{-1}C^{\mathrm{T}} \tag{13.2}$$

Now let $\mathcal{L} = \{A^\alpha C^{\mathrm{T}} : \text{ all } \alpha\}$ and note that $\mathcal{L}$ is reducing for A and contains C^{T}. Thus, by replacing $\mathbb{R}^d$ by $\mathcal{L}$ if necessary, we may assume that C, A_j for $j = 1, \dots, g$ and C^{T} is observable and controllable. It also remains unpinned. This proves the first order growth claim in Theorem 3.3.

To analyze 0^{th} order growth use (13.2) to get

$$(tX) = {}_0 + t\ {}_1(X) + C(I - tL_A(X))^{-1}C^{\mathrm{T}}$$

which has first order growth at ∞ for some $X \in \mathbb{S}^g$ unless ${}_1 = 0$. Thus 0^{th} order growth is equivalent to

$$(X) = {}_0 + C(I - L_A(X))^{-1}C^{\mathrm{T}}$$

as required for our proof. □

We just showed that an NC symmetric rational function $\mathfrak{r}$ with first order growth at infinity, has a symmetric descriptor plus linear realization . In particular, a singularities conclusion for $\mathfrak{r}$ follows from Proposition 9.1. There is no need to consider butterfly representations.

13.2. **Convex Polynomials.**

The fact that convex NC polynomials have degree at most two is a version of the main theorem (symmetric variables case) of Theorem 3.1 in [HM04]. The three proofs below are all very different than that in [HM04]. The starting point for the proofs here is that p has a monic pure minimal butterfly realization,

$$p(x) = r_0 + r_1(x) + \ell(x)\ell(x)^{\mathrm{T}} + L_\Lambda(x)(I - L_A(x))^{-1}L_\Lambda(x)^{\mathrm{T}} \tag{13.3}$$

13.2.1. *Proof Based Upon Growth.* Since p has a monic pure butterfly realization its order of growth at infinity is at most two. If p has degree $m > 2$, then p_m, the homogeneous of degree m part of p is not zero. Thus, using the standard fact that no NC polynomial gives a polynomial identity for matrices of all sizes, see, e.g., [Row80]), there exists a tuple $X \in (\mathbb{S}\mathbb{R}^{n\times n})^g$ so that $p_m(X) \neq 0$; for an elementary proof see [HM04]. It follows that p_m, and hence p, does not have growth of order two at infinity, a contradiction.

13.2.2. *Proof Based on Polynomial Realizations.* This proof is a nearly immediate consequence of the following Lemma.

Lemma 13.1. *In the realization of equation (13.3) (which is a pure minimal butterfly realization) there is an m so that if u is a word of length m, then $A^u = 0$. In particular, since A is symmetric, $A = 0$.*

13.2.3. *Proof Based Upon Singularities Conclusion.* The singularities conclusion of Theorem 3.3 says that minimal pure monic butterfly realizations harbor no hidden singularities. Since p is a polynomial, $I - L_A(x)$ has no zeros. On the other hand, if $A \neq 0$, then, because A is symmetric, there exists an $X \in (\mathbb{SR}^{n\times n})^g$ so that $I - L_A(X)$ has a kernel, a contradiction. Thus $A = 0$ (A is not there).

14. Determinants of Realizations and Determinantal Representations

Representations of polynomials on $\mathbb{C}^2$ and on $\mathbb{R}^2$ as determinants of linear pencils have been studied extensively using line and vector bundles on the (projectivization) of the corresponding plane algebraic curve. Recent articles with lists of references are [V93], [BV96], [BV99], [HVprept], and earlier articles are [D02], [W78], [CT79], [D83], [V89]. However the algebraic-geometrical methods in these papers do not seem to extend to the higher dimensional case.

Here we construct determinantal representations for a symmetric noncommutative polynomial in terms of a symmetric linear pencil. As an immediate consequence this produces a construction of a determinantal representation for every *commutative* polynomial on $\mathbb{R}^g$ in terms of a symmetric linear pencil for any dimension g; not just when $g = 2$. A nonsymmetric determinantal representation of commutative polynomials for any dimension g is due to L. Valiant [Val] (see [BCS] [Chapter 21, especially Section 21.3 and Exercise 21.7] for a good exposition of this and related results); an alternative proof of this result has been communicated to us by Mohan Kumar [Kum].

Theorem 14.1. *Both commutative and noncommutative polynomials have determinantal representations:*

(1) *A polynomial $\check{p}$ on $\mathbb{R}^g$ with $\check{p}(0) \neq 0$ has a symmetric determinantal representation, namely, there are symmetric matrices $J, \widetilde{A}_1, \dots, \widetilde{A}_g$ in $\mathbb{SR}^{d\times d}$ and $J^2 = I$ such that*

$$\check{p}(X) \;=\; const\ \det\left(J - L_{\widetilde{A}}(X)\right) \tag{14.1}$$

for each $X \in \mathbb{R}^g$.

(2) *If p is an NC symmetric polynomial with $p(0) \neq 0$, then there is an NC determinantal representation with symmetric matrices $J, \widetilde{A}_1, \dots, \widetilde{A}_g \in \mathbb{SR}^{d\times d}$ and $J^2 = I$, i.e.,*

$$\det\ p(X) \;=\; const\ \det\left(J \otimes I_n - L_{\widetilde{A}}(X)\right). \tag{14.2}$$

for each $X \in (\mathbb{SR}^{n\times n})^g$ (in fact, each $X \in (\mathbb{R}^{n\times n})^g$).

Warning: d may be larger than the degree of p.

Proof. Result 1 on commutative polynomials follows directly from the results for NC determinantal realizations as we now prove. Suppose $\check{p}$ is a polynomial on $\mathbb{R}^g$. We form a symmetric noncommutative polynomial p which equals $\check{p}$ when restricted to $X \in \mathbb{R}^g$; call p a **noncommutative lift** of $\check{p}$. One way to construct a lift is to replace each (monically normalized) monomial m in $\check{p}$ by a symmetrized word $[x^w + (x^w)^{\mathrm{T}}]/2$ where the word x^w is any lift of m. Now suppose Result 2 of Theorem 14.1 holds and apply it to p. Since for $X \in \mathbb{R}^g$, we know det $p(X) = \check{p}(X)$, the representation (14.2) reduces one such X to representation (14.1) as required.

The proof of item 2 is based on a construction.

14.1. **Direct Algorithm:** Let p be a given symmetric NC polynomial; assume that $p(0) \neq 0$.

Choose a **minimal** symmetric descriptor realization for the symmetric NC polynomial $q = 1 - p$,

$$(x) = C(J - L_A(x))^{-1}C^{\mathrm{T}}.$$

We will prove shortly that

$$\det p(X) = \det(J)\ \det(J - C^{\mathrm{T}}C - L_A(X)). \tag{14.3}$$

Since $p(0) \neq 0$, we have $\det(J - C^{\mathrm{T}}C) \neq 0$ and thus $J - C^{\mathrm{T}}C$ is invertible. Since it is also symmetric, there is another signature matrix $\mathcal{J}$ and a symmetric invertible R so that $R^{-1}\mathcal{J}R^{-1} = J - C^{\mathrm{T}}C$. Then (14.3) yields that

$$\det p(X) = \det(J)\ \det(R^{-2})\ \det(\mathcal{J} - L_{RAR}(X)) \tag{14.4}$$

is a determinantal representation.

Note that A in this construction is always pinned.

Proof that equation (14.3) is true. Let

$$G(x) = \begin{pmatrix} J - L_A(x) & C^{\mathrm{T}} \\ C & 1 \end{pmatrix}$$

Taking Schur complements with respect to the $(1,1)$ entry to produce the LDU decomposition and compute $\det G(X)$ to obtain the left side of

$$\det(J - L_A(X))\ \det(1 - C(J - L_A(X))^{-1}C^{\mathrm{T}}) = \det(J - L_A(X) - C^{\mathrm{T}}C).$$

The right side is gotten similarly from pivoting on $(2,2)$. (To ease the notation we have omitted the tensoring with I_n.)

Since $1 - q = p$, this gives

$$\det(J)\ \det(I - L_{JA}(X))\ \det p(X) \ = \ \det(J - C^{\mathrm{T}}C - L_A(X)).$$

The minimality of the representation for q implies that JA is nilpotent and therefore $\det(I - L_{JA}(x))$ is identically equal to 1, and this gives (14.3). $\square$

The next section presents an alternate algorithm valid up to an unpinned hypothesis.

14.2. **Reciprocal Algorithm:** Suppose p^{-1} is symmetric and denote a symmetric minimal descriptor realization by

$$^{-1}(x) \; = \; \widetilde{D} + \tilde{C}(J - L_{\widetilde{A}}(x))^{-1}\tilde{C}^{\mathrm{T}}.$$

Then, if $\widetilde{A}$ is unpinned,

$$\det p(X) \; = \; \det p(0) \; \det J \; \det \big(J \otimes I_n - L_{\widetilde{A}}(X) \big) . \tag{14.5}$$

for each $X \in (\mathbb{SR}^{n\times n})^g$. (Although it is possible to choose $\tilde{D} = 0$, this may be inconsistent with the unpinned hypothesis below.) The details follow.

The descriptor realization for p^{-1} is easily converted into what is called an FM realization and treated in detail in [BGMprept]. Namely,

(14.6)

$$^{-1}(x) = \widetilde{D}+\tilde{C}J\tilde{C}^{\mathrm{T}}+\tilde{C}(J-L_{\widetilde{A}}(x))^{-1}L_{\widetilde{B}}(x) = \widetilde{D}+\tilde{C}J\tilde{C}^{\mathrm{T}}+\tilde{C}(I-L_{J\widetilde{A}}(x))^{-1}L_{J\widetilde{B}}(x),$$

where $\widetilde{B}_j = \widetilde{A}_j J\tilde{C}^T$. For notational ease, let now $D = \widetilde{D} + \tilde{C}J\tilde{C}^{\mathrm{T}}$, $C = \tilde{C}$, $A_j = J\widetilde{A}_j$ and $B = JA_jJC^T$. Note that the A_j are not necessarily symmetric. The FM realization $D+C(I-L_A(x))^{-1}L_B(x)$ will be minimal in the sense that both $\{(JA)^w B_j : w, j\}$ and $\{(AJ)^w C : w\}$ span $\mathbb{R}^d$ if the original descriptor realization was unpinned which we now assume.

The inverse (see [BGMprept]) of the minimal FM realization in equation (14.6) is a minimal FM realization for p,

$$(x) = D^{-1} - D^{-1}C(I - L_{A^\times}(x))^{-1}L_B(x)D^{-1}, \tag{14.7}$$

where $A_j^\times := A_j - BD^{-1}C$. Since the realization in equation (14.7) is minimal and represents a polynomial, $(A^\times)^w = 0$ for words w of sufficiently long length (see the proof of Lemma 13.1). In particular, $L_{A^\times}(x)$ is nilpotent matrix valued function.

For $X \in (\mathbb{SR}^{n\times n})^g$, a Schur complement calculation applied alternatively to the (1,1) and the (2,2) entry of the matrix

$$G := \begin{bmatrix} L_A(X) - I_{nd} & \sum_j (B_j \otimes X_j) \\ C \otimes I_n & D \otimes I_n \end{bmatrix}.$$

yields

$$\det \left(L_{A^\times}(X) - I_{nd}\right) \, \det(D \otimes I_n) \; = \; \det p(X)^{-1} \, \det \left(L_A(X) - I_{nd}\right). \tag{14.8}$$

Since $L_{A^\times}(X)$ is nilpotent, the left hand side is constant and the desired representation follows.

Example 14.2. Let $q = 2 + x^2$. A minimal symmetric descriptor realization is gotten by choosing

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C^{\mathrm{T}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

In this case

$$JA = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad JC^{\mathrm{T}} = C^{\mathrm{T}}.$$

Further,

$$(I - JAx)^{-1} = \begin{pmatrix} 1 & -x & 0 \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & x & x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus,

$$C(I - JAx)^{-1} JC^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2 + x^2$$

as claimed.

Next,

$$J - C^{\mathrm{T}} C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In particular, $J - C^{\mathrm{T}} C$ is once again already a similarity and thus $\mathcal{J} = J - C^{\mathrm{T}} C$ and $R^{-1} A R^{-1} = A$.

Direct computation gives,

$$\begin{aligned} \det(J) \det(J - C^{\mathrm{T}} C - Ax) &= -\det \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -x \\ 0 & -x & -1 \end{pmatrix} \\ &= -1 - x^2 = 1 - q = p. \end{aligned}$$

$\square$

14.3. **Open Questions.** 1. Find an algorithm which will produce a monic determinantal representation if one exists.

2. We conjecture that the Reciprocal Algorithm always works, i.e., for any NC (symmetric) polynomial p (with $p(0) \neq 0$), p^{-1} admits a minimal *unpinned* (symmetric) descriptor realization; equivalently, the minimal (symmetric) FM realization of p^{-1} is necessarily unpinned. Evidence comes from John Shopple who has run many examples with his implementation (under Mathematica) of Slinglend's Algorithm [Sprept] for producing descriptor realizations.

15. Linear System Theory Motivation

Matrix inequalities (MIs) have come to be extremely important in linear systems engineering in the past decade. This is because many linear systems problems convert directly into matrix inequalities.

Matrix inequalities take the form of a list of requirements that polynomials or rational functions of matrices or matrices containing rational functions of matrices be positive semidefinite. Of course while some engineering problems present rational functions which are well behaved, many other problems present rational functions which are badly behaved. Thus taking the list of functions which a design problem presents and converting these to a nice form, or at least checking if they already have or do not have a nice form is a major enterprise. Since matrix multiplication is not commutative, one sees much effort going into calculations (by hand) on noncommutative rational functions, although engineers seldom use (don't like even) the word noncommutative. A major goal in systems engineering is to convert, if possible,"noncommutative inequalities" to equivalent linear noncommutative inequalities (effectively to LMIs).

A simple example of nicely behaved MIs is the Riccati inequality

$$ax + xa^{\mathrm{T}} - xbb^{\mathrm{T}}x + c^{\mathrm{T}}c \ \ \textit{is positive semidefinite} \tag{15.1}$$

Also there is the LMI

$$\begin{pmatrix} ax + xa^{\mathrm{T}} + c^{\mathrm{T}}c & xb \\ b^{\mathrm{T}} & I \end{pmatrix} \ \ \textit{is positive semidefinite.} \tag{15.2}$$

The inequalities (15.1) and (15.2) are equivalent in that given matrices A, B, C they have the same set of solutions X. Note (15.2) is linear in the unknown X; thus is an LMI. It is algebraic formulas like these (though typically more complicated) that are programmed into the main computer packages in engineering.

A user of one of these packages when doing a design puts in the math model for his system, that is, he gives specific matrices A, B, C. Numerical software in the package then solves for X.

Thus to produce design software there are two main issues.
(1) Algebraic: complicated inequalities involving polynomials and rational functions occur, convert them to nice ones or prove this impossible.
(2) Numerical: Find numerical methods for solving nice ones.

Convexity is a major issue because ultimately numerical methods called semidefinite programming are optimization based. LMI's play a dominant role now in systems algorithms and software; at least a thousand papers concern them. The state of the engineering art is: there are clever tricks for producing LMI, but little that is systematic and in many problems MIs but no LMIs emerge. The paper aims at the beginnings of an algebraic theory which might be helpful for determining which MIs convert to LMIs and how this conversion might be done automatically.

15.1. **To Commute or Not Commute: "Dimensionless" Formulas.** This section discusses two different ways of writing matrix inequalities. It follows [H03]. As an example, we could consider either the Riccati inequality (15.1) or the equivalent LMI in (15.2). Let us focus on this LMI, and discuss the various ways one could write this linear matrix inequality.

The LMI in (15.2) has the same form regardless of the dimension of the system and its defining matrices A, B, C. In other words, if we take the matrices A, B, C and X to have compatible dimension, (regardless of what those dimensions are), then the inequality (15.2) is meaningful and substantive and its form does not change.

When the dimensions of the matrices A, B, C and X are specified it is common to write (15.2) as a linear combination of known matrices $L_0, L_1, \ldots, L_g$ of dimension $d \times d$ in unknown real numbers $s_1, \ldots, s_g$:

$$L_0 + \sum_{j=1}^{g} L_j s_j \text{ is positive semidefinite} \tag{15.3}$$

For example, in the inequality (15.3) if $A \in \mathbb{R}^{2\times 2}, B \in \mathbb{R}^{2\times 1}, C \in \mathbb{R}^{1\times 2}$, then $X^{\mathrm{T}} = X \in \mathbb{R}^{2\times 2}$ and we would take $m = 3$ and the numbers s_i in $X = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$ as unknowns in the inequality (15.3). The unpleasant part is that the L_i are $L_0 := \begin{pmatrix} C^{\mathrm{T}}C & 0 \\ 0 & I \end{pmatrix}$ $\quad L_1 := \begin{pmatrix} 2a_{11} & a_{21} & b_{11} & b_{12} \\ a_{21} & 0 & 0 & 0 \\ b_{11} & 0 & 0 & 0 \\ b_{12} & 0 & 0 & 0 \end{pmatrix}$

$$L_2 := \begin{pmatrix} 2a_{12} & a_{11}+a_{22} & b_{21} & b_{22} \\ a_{22}+a_{11} & 2a_{21} & b_{11} & b_{12} \\ b_{21} & b_{11} & 0 & 0 \\ b_{22} & b_{12} & 0 & 0 \end{pmatrix} \quad L_3 := \begin{pmatrix} 0 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & a_{22} \\ 0 & 0 & 0 & b_{21} \\ a_{12} & a_{22} & b_{21} & 2b_{22} \end{pmatrix}.$$

Now consider $A \in \mathbb{R}^{3\times 3}, B \in \mathbb{R}^{3\times 2}, C \in \mathbb{R}^{2\times 3}, X \in \mathbb{R}^{3\times 3}$. This gives a messier formula. The point is that the formula (15.3), with commutative unknowns, does not scale simply with dimension of the matrices or of the system producing them, while formula (15.2) does, but (15.2) contains noncommutative unknowns.

Problems are split into two natural types: **dimensionless**, the dimension of the system does not directly enter the statement of the problem, and **dimension dependent**. Often one sees this in problems where the diagram of systems interconnections is specified. Most classical systems problems are dimensionless, e.g. the classical H^2 control problem, H^∞ control problem, state estimation problems, etc.. It is an empirical observation that dimensionless problems convert to matrix inequalities in noncommutative variables, while those which are dimension dependent lose this structure and have commutative variables. For example, the H^2 control problem converts to solving one Riccati inequality, while the H^∞ control problem converts to solving two Riccati's and a coupling inequality; all of these are inequalities on polynomials in noncommutative variables.

15.2. **Open Questions.** Two questions arise if one aims to extend the results of this paper to the level of generality seen in these engineering examples.

One needs to extend our main theorems to matrix valued rational functions. Many of the arguments here in fact go through directly to matrix valued functions.

The other problem is to generalize the main results to rational functions whose coefficients are indeterminates or combinations of them. This may be formidable but [CHSY03] treats this type of rational function successfully and many of the techniques apply here. Slinglend's algorithm works at this level of generality except the aspect of cutting a given realization down to a minimal one is problematic.

16. Appendix: Noncommutative Rational Functions

Here we add more detail to §2.2 on the class of NC rational functions used in this paper. Recall the main issue, which is quite familiar to the expert, is that we wish to deal with rational expressions, like $r = x_1(1 - x_2x_1)^{-1}$ but there are other rational expressions like $r_2 = (1 - x_1x_2)^{-1}x_1$ for the "same" rational function. Thus one needs to specify an equivalence relation on rational expressions. There are various frameworks for this, see [L03] for a survey. The one we use here uses rational former power series on the one hand (see the book [BR84] or articles [S61], [F74a], [F74b]) and on the other hand rational expressions familiar in the theory of rings with rational identities (see, e.g., [Row80, Chapter 8]). We include this appendix because although the notions are well known, the precise framework as we need it, with a special emphasis on matrix substitutions and domains of definitions, does not seem to be laid out elsewhere explicitly. We wish to thank Lance Small for valuable discussions.

Notice that the base field $\mathbb{R}$ can be replaced everywhere by $\mathbb{C}$.

16.1. **Noncommutative Rational Expressions.** We define recursively the notions of a **noncommutative rational expression** r **in** $x_1, \dots, x_g$ **analytic at zero** and its **value at zero** $r(0)$. We also define the **formal domain** $\mathcal{NF}(n)_{r,\mathbf{for}}$ of r on g-tuples of $n \times n$ matrices (which will be a non empty Zariski open subset of $(\mathbb{R}^{n\times n})^g$ containing $(0, \dots, 0)$), and the **evaluation**

$$r(X_1, \dots, X_g) \in \mathbb{R}^{n\times n} \quad for\ (X_1, \dots, X_g) \in \mathcal{NF}(n)_{r,\mathrm{for}}.$$

Definition 16.1. (1) If p is in $\mathbb{R}\langle x_1, \dots, x_g\rangle$, that is, if p is a polynomial, then p is a noncommutative rational expression analytic at zero; $p(0)$ is the constant coefficient of p. $\mathcal{NF}(n)_{p,\mathrm{for}} = (\mathbb{R}^{n\times n})^g$ and the evaluation $p(X_1, \dots, X_g)$ is defined in the obvious way described in §2.2.

(2) If r_1 and r_2 are noncommutative rational expressions analytic at zero, then $r_1 + r_2$, r_1r_2 are also noncommutative rational expression analytic at zero;

$$(r_1 + r_2)(0) = r_1(0) + r_2(0),\ (r_1r_2)(0) = r_1(0)r_2(0),$$
$$\mathcal{NF}(n)_{r_1+r_2,\mathrm{for}} = \mathcal{NF}(n)_{r_1,\mathrm{for}} \cap \mathcal{NF}(n)_{r_2,\mathrm{for}},$$
$$\mathcal{NF}(n)_{r_1r_2,\mathrm{for}} = \mathcal{NF}(n)_{r_1,\mathrm{for}} \cap \mathcal{NF}(n)_{r_2,\mathrm{for}}$$

and the evaluation satisfies

$$(r_1 + r_2)(X_1, \dots, X_g) = r_1(X_1, \dots, X_g) + r_2(X_1, \dots, X_g),$$
$$(r_1r_2)(X_1, \dots, X_g) = r_1(X_1, \dots, X_g)r_2(X_1, \dots, X_g).$$

(3) If r is a noncommutative rational expression analytic at zero and $r(0) \neq 0$, then r^{-1} is also a noncommutative rational expression analytic at zero; $(r^{-1})(0) = (r(0))^{-1}$.

$$\mathcal{NF}(n)_{r^{-1},\mathrm{for}} = \{(X_1, \dots, X_g) \in \mathcal{NF}(n)_{r,\mathrm{for}} \colon\ \det r(X_1, \dots, X_g) \neq 0\}$$

and

$$(r^{-1})(X_1, \dots, X_g) = (r(X_1, \dots, X_g))^{-1}.$$

Remark 16.2. Obviously, $r(0, \dots, 0) = r(0)I_n$.

Remark 16.3. Now we list a convenient fact which follows from the definition. If r is a noncommutative rational expression analytic at zero, then $-r = -1 \cdot r$ is also a noncommutative rational expression analytic at zero; $(-r)(0) = -r(0)$. $\mathcal{NF}(n)_{-r,\text{for}} = \mathcal{NF}(n)_{r,\text{for}}$ and the evaluation satisfies $(-r)(X_1, \dots, X_g) = -r(X_1, \dots, X_g)$.

Remark 16.4. It is obvious that a noncommutative rational expression analytic at zero r defines a rational function $\mathtt{r}$ on $(\mathbb{R}^{n\times n})^g \cong \mathbb{R}^{gn^2}$ with values in $\mathbb{R}^{n\times n}$ for every n with a domain of analyticity containing $\mathcal{NF}(n)_{r,\text{for}}$.

At several points we need the following technical lemma.

Lemma 16.5. *Let r be a noncommutative rational expression analytic at zero. Then for every $\epsilon > 0$ there exists $\delta = \delta_r(\epsilon) > 0$ such that if $(X_1, \dots, X_g) \in (\mathbb{R}^{n\times n})^g$ with $\|X_i\| < \delta$ for all i, then $(X_1, \dots, X_g) \in \mathcal{NF}(n)_{r,for}$ and*

$$\|r(X_1, \dots, X_g) - r(0, \dots, 0)\| < \epsilon.$$

The point is of course that δ is independent of n.

Proof of Lemma 16.5. We shall prove the result by recursion following Definition 16.1.

(1) Let $r = p = \sum_{|w|\le m} p_w x^w$ be a noncommutative polynomial of degree m. Given $\epsilon > 0$, choose $\delta > 0$ so that

$$\sum_{0<|w|\le m} |p_w|\delta^{|w|} < \epsilon.$$

Then for any $(X_1, \dots, X_g) \in (\mathbb{R}^{n\times n})^g$ with $\|X_i\| < \delta$ for all i,

$$\|p(X_1, \dots, X_g) - p(0, \dots, 0)\| = \|\sum_{0<|w|\le m} p_w X^w\| \le \sum_{0<|w|\le m} |p_w|\delta^{|w|} < \epsilon.$$

(2) Let r_1 and r_2 be noncommutative expressions analytic at zero. Given $\epsilon > 0$, take $\epsilon' > 0$ so that $\epsilon' \le \epsilon/2$, and let $\delta_1 = \delta_{r_1}(\epsilon')$, $\delta_2 = \delta_{r_2}(\epsilon')$. Then for $(X_1, \dots, X_g) \in (\mathbb{R}^{n\times n})^g$ with $\|X_i\| < \min(\delta_1, \delta_2)$ for all i, we have that $(X_1, \dots, X_g) \in \mathcal{NF}(n)_{r_1+r_2,\text{for}}$ and

$$\begin{aligned}
&\|(r_1 + r_2)(X_1, \dots, X_g) - (r_1 + r_2)(0, \dots, 0)\| \\
&\quad \le \|r_1(X_1, \dots, X_g) - r_1(0, \dots, 0)\| + \|r_2(X_1, \dots, X_g) - r_2(0, \dots, 0)\| \\
&\qquad < \epsilon' + \epsilon' \le \epsilon.
\end{aligned}$$

Similarly, given $\epsilon > 0$, take $\epsilon' > 0$ so that

$$(\epsilon')^2 \le \epsilon/3, |r_1(0)|\epsilon' \le \epsilon/3, |r_2(0)|\epsilon' \le \epsilon/3,$$

and let $\delta_1 = \delta_{r_1}(\epsilon')$, $\delta_2 = \delta_{r_2}(\epsilon')$. Then for $(X_1, \ldots, X_g) \in (\mathbb{R}^{n\times n})^g$ with $\|X_i\| < \min(\delta_1, \delta_2)$ for all i, we have $(X_1, \ldots, X_g) \in \mathcal{NF}(n)_{r_1 r_2, \text{for}}$ and

$$\begin{aligned}
&\|(r_1 r_2)(X_1, \ldots, X_g) - (r_1 r_2)(0, \ldots, 0)\| \\
&\quad \le \|r_1(X_1, \ldots, X_g) - r_1(0, \ldots, 0)\| \, \|r_2(X_1, \ldots, X_g) - r_2(0, \ldots, 0)\| \\
&\qquad + \|r_1(0, \ldots, 0)\| \, \|r_2(X_1, \ldots, X_g) - r_2(0, \ldots, 0)\| \\
&\qquad + \|r_1(X_1, \ldots, X_g) - r_1(0, \ldots, 0)\| \, \|r_2(0, \ldots, 0)\| \\
&\qquad\qquad < (\epsilon')^2 + |r_1(0)|\epsilon' + |r_2(0)|\epsilon' \le \epsilon.
\end{aligned}$$

(3) Finally, let r be a noncommutative rational expression analytic at zero with $r(0) \neq 0$. Given $\epsilon > 0$, take $\epsilon' > 0$ so that $|r(0)^{-1}\epsilon'| < 1$, $(1 - |r(0)^{-1}|\epsilon')^{-1}\epsilon' \le \epsilon$, and let $\delta = \delta_r(\epsilon')$. Then for $(X_1, \ldots, X_g) \in (\mathbb{R}^{n\times n})^g$ with $\|X_i\| < \delta$ for all i we have $(X_1, \ldots, X_g) \in \mathcal{NF}(n)_{r, \text{for}}$ and

$$r(X_1, \ldots, X_g) = r(0, \ldots, 0)(I_n - r(0, \ldots, 0)^{-1}(r(0, \ldots, 0) - r(X_1, \ldots, X_g)));$$

since

$$\|r(0, \ldots, 0)^{-1}(r(X_1, \ldots, X_g) - r(0, \ldots, 0))\| \le |r(0)^{-1}|\epsilon' < 1,$$

it follows that $r(X_1, \ldots, X_g)$ is invertible and hence $(X_1, \ldots, X_g) \in \mathcal{NF}(n)_{r^{-1}, \text{for}}$; furthermore

$$\begin{aligned}
&\|r(X_1, \ldots, X_g)^{-1}\| \\
&\quad \le \|r(0, \ldots, 0)\|(1 - \|r(0, \ldots, 0)^{-1}(r(0, \ldots, 0) - r(X_1, \ldots, X_g))\|)^{-1} \\
&\qquad\qquad < |r(0)|(1 - |r(0)^{-1}|\epsilon')^{-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|(r^{-1})(X_1, \ldots, X_g) - (r^{-1})(0, \ldots, 0)\| \\
&\quad \le \|r(X_1, \ldots, X_g)^{-1}\| \, \|r(X_1, \ldots, X_g) - r(0, \ldots, 0)\| \, \|r(0, \ldots, 0)^{-1}\| \\
&\qquad\qquad < |r(0)|(1 - |r(0)^{-1}|\epsilon')^{-1}\epsilon'|r(0)^{-1}| \le \epsilon.
\end{aligned}$$

□

16.2. **Rational Noncommutative Formal Power Series.** We assume the reader has some experience with formal power series

$$\sum_{w \in \mathcal{W}_g} r_w x^w$$

and denote by $\mathbb{R}\langle\langle x_1, \ldots, x_g\rangle\rangle$ = the ring of noncommutative formal power series over $\mathbb{R}$ in g noncommuting variables $x_1, \ldots, x_g$.

As an example, we consider the operation of inversion. If p is a NC polynomial write $p = p(0) + q$ where $q(0) = 0$, then the inverse $r = p^{-1}$ is the series expansion $r = \frac{1}{p(0)} \sum_k q^k$. Note on a small enough Y in $(\mathbb{R}^{n\times n})^g$ the series for r is convergent on Y.

Definition 16.6. The ring $\mathbb{R}\langle\langle x_1, \ldots, x_g\rangle\rangle_{\text{rat}}$ of **rational noncommutative formal power series** is the smallest subring of $\mathbb{R}\langle\langle x_1, \ldots, x_g\rangle\rangle$ containing the noncommutative polynomials and closed under inversion (of invertible elements).

It is obvious that any noncommutative rational expression analytic at zero determines a rational noncommutative formal power series (which then necessarily converges on some neighborhood of $(0,\dots,0)$ in $(\mathbb{R}^{n\times n})^g$), and any rational noncommutative formal power series can be obtained in this way.

Proposition 16.7. *For two noncommutative rational expressions analytic at zero, r_1 and r_2, the following are equivalent:*

1. *r_1 and r_2 determine the same formal power series.*
2. *If n is a positive integer, then r_1 and r_2 define the same analytic function on a neighborhood of $(0,\dots,0)$ in $(\mathbb{R}^{n\times n})^g$.*
3. *If n is a positive integer, then r_1 and r_2 define the same analytic function on an open set in $(\mathbb{R}^{n\times n})^g$.*
4. *If n is a positive integer, then r_1 and r_2 define the same $\mathbb{R}^{n\times n}$-valued rational function on $(\mathbb{R}^{n\times n})^g$.*
5. *items 2–4 hold with $(\mathbb{R}^{n\times n})^g$ replaced by $(\mathbb{SR}^{n\times n})^g$*

Proof. The fact that 2–4 are equivalent follows from standard properties of rational functions. It is obvious that 1 implies 2. The fact that 2 implies 1 follows from the identity theorem for convergent noncommutative formal power series (the fact that if f is a noncommutative formal power series which converges in a neighborhood of $(0,\dots,0)$ in $(\mathbb{R}^{n\times n})^g$ for every n and vanishes there identically, then $f=0$) which follows easily from the standard identity theorem for noncommutative polynomials by separating homogeneous terms of different degrees.

To prove that 5 is equivalent to 1–4, it is enough to show that a convergent noncommutative formal power series that vanishes on g-tuples of symmetric matrices, vanishes identically. Separating homogeneous terms of different degrees, we see that it is enough to prove the following: let $p(x_1,\dots,x_g)=\sum_{w\in\mathcal{W}_g,|w|=k} p_w x^w$ be a homogeneous noncommutative polynomial of degree k; if $p(S_1,\dots,S_g)=0$ for any $(S_1,\dots,S_g)\in((\mathbb{SR}^{n\times n})^g)^g$, then $p_w=0$ for all w.

For arbitrary $(X_1,\dots,X_g)\in(\mathbb{R}^{n\times n})^g$, let us define $(S_1,\dots,S_g)\in(\mathbb{SR}^{2n\times 2n})^g$ by

$$S_j=\begin{bmatrix}0 & X_j\\ X_j^{\mathrm{T}} & 0\end{bmatrix}.$$

Assume first that k is even; if $w=\chi_{i_1}\dots\chi_{i_k}$ then

$$S^w=\begin{bmatrix}X_{i_1}X_{i_2}^{\mathrm{T}}\dots X_{i_{k-1}}X_{i_k}^{\mathrm{T}} & 0\\ 0 & X_{i_1}^{\mathrm{T}}X_{i_2}\dots X_{i_{k-1}}^{\mathrm{T}}X_{i_k}\end{bmatrix}.$$

Since $p(S_1,\dots,S_g)=0$ we conclude that the noncommutative polynomial in x and x^{T} given by

$$\sum_{w\in\mathcal{W}_g,|w|=k\colon\, w=\chi_{i_1}\cdots\chi_{i_k}} p_w x_{i_1}x_{i_2}^{\mathrm{T}}\dots x_{i_{k-1}}x_{i_k}^{\mathrm{T}}$$

vanishes identically on $(\mathbb{R}^{n\times n})^g$ for any n. Taking $n > k$ gives $p_w = 0$ for all w (the ring $\mathbb{R}^{n\times n}$ with transposition satisfies no polynomial identities with involution of degree less than n, see [Row80, Remark 2.5.14]).

The case k is odd is treated similarly; for any word $w = \chi_{i_1} \dots \chi_{i_k}$ we have

$$S^w = \begin{bmatrix} 0 & X_{i_1} X_{i_2}^{\mathrm{T}} \dots X_{i_{k-2}} X_{i_{k-1}}^{\mathrm{T}} X_{i_k} \\ X_{i_1}^{\mathrm{T}} X_{i_2} \dots X_{i_{k-2}}^{\mathrm{T}} X_{i_{k-1}} X_{i_k}^{\mathrm{T}} & 0 \end{bmatrix}.$$

Since $p(S_1, \dots, S_g) = 0$ we conclude that noncommutative polynomial in x and x^{T} given by

$$\sum_{w\in\mathcal{W}_g, |w|=k:\ w=\chi_{i_1}\cdots\chi_{i_k}} p_w x_{i_1} x_{i_2}^{\mathrm{T}} \dots x_{i_{k-2}} x_{i_{k-1}}^{\mathrm{T}} x_{i_k}$$

vanishes identically on $(\mathbb{R}^{n\times n})^g$ for any n. As before, taking $n > k$ gives $p_w = 0$ for all w. □

16.3. **Noncommutative Rational Functions.** The formal series expansions define a natural equivalence on NC rational expressions.

Definition 16.8. A **noncommutative rational function analytic at zero** is an equivalence class $\mathfrak{r}$ of noncommutative rational expressions analytic at zero under the equivalence relation given by the equivalent conditions 1–5 of Proposition 16.7.

For an example see (2.1) and (2.2).

The ring $\mathbb{R}\langle x\rangle_{Rat0}$ of noncommutative rational functions analytic at zero is thus isomorphic to the ring of rational noncommutative formal power series $\mathbb{R}\langle\langle x_1, \dots, x_g\rangle\rangle_{\mathrm{rat}}$.

It follows from Definitions 16.1 and 16.8 that for any noncommutative rational function analytic at zero, $\mathfrak{r}$, *we may define uniquely a noncommutative rational function analytic at zero,* $\mathfrak{r}^{\mathrm{T}}$*, such that*

$$\left(\mathfrak{r}^{\mathrm{T}}(X_1^{\mathrm{T}}, \dots, X_g^{\mathrm{T}})\right)^{\mathrm{T}} = \mathfrak{r}(X_1, \dots, X_g).$$

It follows from the identity theorem for noncommutative formal power series that if

$$\mathfrak{r}(X_1, \dots, X_g) = \sum_{w\in\mathcal{W}_g} \mathfrak{r}_w x^w$$

then

$$\mathfrak{r}^{\mathrm{T}}(X_1, \dots, X_g) = \sum_{w\in\mathcal{W}_g} \mathfrak{r}_{w^{\mathrm{T}}} x^w.$$

16.4. **Matrices of Rational Expressions and Rational Functions, and Matrix Valued Rational Expressions and Rational Functions.** Now we turn to the matrix case of what we just finished.

16.4.1. *Matrices of Rational Expressions and Rational Functions.* We first notice the following fact.

Proposition 16.9. *If $\mathfrak{R}$ is a $d \times d$ matrix of noncommutative rational functions analytic at zero and $\mathfrak{R}(0)$ is an invertible matrix, then $\mathfrak{R}$ is invertible in $(\mathbb{R}\langle x\rangle_{Rat0})^{d\times d}$.*

Proof. This follows immediately from the fact that

$$T(x_1,\dots,x_g) = \sum_{w\in\mathcal{W}_g} T_w x^w \in (\mathbb{R}\langle\langle x_1,\dots,x_g\rangle\rangle)^{d_1\times d_2}$$

belongs to $(\mathbb{R}\langle\langle x_1,\dots,x_g\rangle\rangle_{\mathrm{rat}})^{d_1\times d_2}$ if and only if it admits a noncommutative Fornasini–Marchesini (FM) realization as in [BGMprept], that is,

$$T_{w\chi_j} = CA^w B_j$$

for some matrices C, $A_1,\dots,A_g$, $B_1,\dots,B_g$, and from the inversion formula for FM realizations.

We shall give an alternative direct proof by showing how to construct explicitly a matrix Q of rational expressions representing $\mathfrak{R}^{-1}$.

Let R be a matrix of noncommutative rational expressions analytic at zero, with $R(0)$ invertible. Multiplying $R(0)$ from the left and from the right by appropriate permutation matrices E and F, we have an LDU decomposition, $ER(0)F = L_0D_0U_0$, where L_0 and U_0 are respectively lower and upper triangular matrices with ones on the main diagonal and D_0 is a block diagonal matrix with nonzero diagonal elements or 2×2 blocks on the diagonal whose diagonal entries are 0 and off diagonal entries are 0.

We have now selected permutations E, F and we now apply the algebraic LDU decomposition of [CHSY03] to ERF. We obtain that ERF is equivalent entry-wise to LDU where L, D, and U are matrices of rational expressions with L and U respectively lower and upper triangular with ones on the main diagonal and D block diagonal with $D(0) = D_0$. Furthermore, the entries of L, D, and U are obtained from the entries of R using addition, multiplication and inversion, with the only expressions inverted, the so called “pivots”, being the first $d-1$ diagonal or block diagonal entries of D, thus the entries are rational expressions. The only problem which can arise with this decomposition is that a pivot not be 0 at 0, however, E, F were chosen to insure this does not happen.

It follows that the inverse $\mathfrak{Q}$ of $\mathfrak{R}$ in $(\mathbb{R}\langle x_1,\dots,x_g\rangle_0)^{d\times d}$ is represented by a matrix Q of rational expressions, where $Q = E^{-1}U^{-1}D^{-1}L^{-1}F^{-1}$. Here U^{-1}, D^{-1} and Q^{-1} have an obvious meaning: D^{-1} is the diagonal matrix obtained from D by inverting the entries, $L^{-1} = I + (I-L) + \ldots + (I-L)^{d-1}$, $U^{-1} = I + (I-U) + \ldots + (I-U)^{d-1}$. ∎

Now we give an example illustrating the construction used in the proof. Let $R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22}\end{pmatrix}$ with $\det R(0) \neq 0$. Assume that $r_{11}(0) \neq 0$. Then an easy calculation shows $\mathfrak{R}^{-1}$ is represented by

$$\begin{pmatrix} 1 & -r_{11}^{-1}r_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_{11}^{-1} & 0 \\ 0 & (r_{22} - r_{21}r_{11}^{-1}r_{12})^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -r_{21}r_{11}^{-1} & 1 \end{pmatrix}$$

(the fact that $r_{22}(0) - r_{21}(0)r_{11}(0)^{-1}r_{12}(0) \neq 0$ is of course implied by $r_{11}(0) \neq 0$ and $\det R(0) \neq 0$).

16.4.2. *Matrix Valued NC Rational Expressions.* We define a $d_1 \times d_2$**-matrix valued noncommutative rational expression** R **in** $x_1, \dots, x_g$ **analytic at zero** and its domain $\mathcal{NF}(n)_{R,\mathrm{for}}$ as in Definition 16.1 except that we start with noncommutative polynomials with coefficients in $\mathbb{R}^{d_1 \times d_2}$ and use matrix operations whenever these make sense:

(1) $P \in \mathbb{R}^{d_1 \times d_2}\langle x_1, \dots, x_g \rangle$ is a $d_1 \times d_2$-matrix valued noncommutative rational expression analytic at zero, $P(0)$ is the constant term of P. $\mathcal{NF}(n)_{P,\mathrm{for}} = (\mathbb{R}^{n\times n})^g$ and the evaluation $P(X_1, \dots, X_g)$ is defined using tensor substitution of matrices in a noncommutative polynomial with matrix coefficients as explained in Section 2.1.2.

(2) If R_1 and R_2 are $d_1 \times d_2$-matrix valued noncommutative rational expressions analytic at zero, then $R_1 + R_2$ is a $d_1 \times d_2$-matrix valued noncommutative rational expression analytic at zero,
$$(R_1 + R_2)(0) = R_1(0) + R_2(0);$$
if R_1 and R_2 are a $d_1 \times d'$-matrix valued and a $d' \times d_2$-matrix valued noncommutative rational expressions analytic at zero, respectively, then R_1R_2 is a $d_1 \times d_2$-matrix valued noncommutative rational expression analytic at zero, $(R_1R_2)(0) = R_1(0)R_2(0)$.
$$\mathcal{NF}(n)_{R_1+R_2,\mathrm{for}} = \mathcal{NF}(n)_{R_1,\mathrm{for}} \cap \mathcal{NF}(n)_{R_2,\mathrm{for}},$$
$$\mathcal{NF}(n)_{R_1R_2,\mathrm{for}} = \mathcal{NF}(n)_{R_1,\mathrm{for}} \cap \mathcal{NF}(n)_{R_2,\mathrm{for}}$$
and the evaluation satisfies
$$(R_1 + R_2)(X_1, \dots, X_g) = R_1(X_1, \dots, X_g) + R_2(X_1, \dots, X_g),$$
$$(R_1R_2)(X_1, \dots, X_g) = R_1(X_1, \dots, X_g)R_2(X_1, \dots, X_g).$$

(3) If R is a $d \times d$-matrix valued noncommutative rational expression analytic at zero and $R(0) \in \mathbb{R}^{d\times d}$ is invertible, then R^{-1} is a $d \times d$-matrix valued noncommutative rational expression analytic at zero, $(R^{-1})(0) = R(0)^{-1}$.
$$\mathcal{NF}(n)_{R^{-1},\mathrm{for}} = \{(X_1, \dots, X_g) \in \mathcal{NF}(n)_{R,\mathrm{for}} \colon \det R(X_1, \dots, X_g) \neq 0\}$$
and
$$(R^{-1})(X_1, \dots, X_g) = (R(X_1, \dots, X_g))^{-1}.$$

It is obvious that all the results of Sections 16.1–16.3 hold for matrix valued rational expressions, with obvious modifications: a $d_1 \times d_2$-matrix valued noncommutative rational expression analytic at zero R determines a rational noncommutative formal power series with coefficients in $\mathbb{R}^{d_1\times d_2}$, and a rational function on $(\mathbb{R}^{n\times n})^g \cong \mathbb{R}^{gn^2}$ with values in $\mathbb{R}^{d_1 n \times d_2 n}$ for every n with a domain of analyticity containing $\mathcal{NF}(n)_{R,\mathrm{for}}$.

As in Section 16.3, we define a $d_1 \times d_2$**-matrix valued noncommutative rational function analytic at zero** to be an equivalence class of $d_1 \times d_2$-matrix valued noncommutative rational expressions analytic at zero. When there is lack of clarity, we shall refer to noncommutative rational expressions and functions considered in Sections 16.1 – 16.3 as **scalar** rational expressions and functions.

Notice that a 1×1-matrix valued noncommutative rational expression is more general than a scalar noncommutative rational expression since we may use matrices in the process, see, e.g., Example 2.5. On the other hand, Proposition 16.9 implies that a $d_1 \times d_2$-matrix valued noncommutative rational function analytic at zero is the same as a $d_1 \times d_2$ matrix of scalar noncommutative rational functions analytic at zero. (Equivalently, as noticed in the proof of Proposition 16.9, a rational noncommutative formal power series with coefficients in $\mathbb{R}^{d_1 \times d_2}$ is the same as a $d_1 \times d_2$ matrix of scalar rational noncommutative formal power series.) In other words, every $d_1 \times d_2$-matrix valued noncommutative rational function analytic at zero can be represented by a $d_1 \times d_2$ matrix of scalar noncommutative rational expressions analytic at zero.

16.4.3. *Two Notions of Domains.* Now we compare the domain of a matrix of scalar NC rational expressions to the domain of an equivalent matrix valued NC rational expression.

Theorem 16.10. *For $\mathfrak{R}$ a $d_1 \times d_2$-matrix valued noncommutative rational function analytic at zero, the domain $\mathcal{F}^{mr}_{\mathfrak{R},for}$ defined by*

$$\bigcup_{R=[r_{ij}]_{i=1,\dots,d_1;j=1,\dots,d_2} \text{ is a } d_1 \times d_2 \text{ matrix of scalar NC rational expressions in } \mathfrak{R}} \cap_{i,j} \mathcal{NF}(n)_{r_{ij},for}$$

is the same as the domain $\mathcal{F}^{mvr}_{\mathfrak{R},for}$ defined by

$$\bigcup_{R \text{ is a } d_1 \times d_2\text{-matrix valued NC rational expression in } \mathfrak{R}} \mathcal{NF}(n)_{R,for}.$$

The inclusion

$$\mathcal{F}^{mr}_{\mathfrak{R},for} \subset \mathcal{F}^{mvr}_{\mathfrak{R},for}$$

is obvious. What requires a proof is the reverse inclusion and for this we use the following proposition.

Proposition 16.11. *Suppose we are given a $m \times m$ matrix of scalar NC rational expression R whose constant term $R(0)$ is invertible. If $X \in (\mathbb{SR}^{n \times n})^g$ and $R(X)$ is invertible, then there exists a matrix Q of scalar NC rational expressions such that $QR = I$ and X is in the domain of Q. Moreover, if R is symmetric, then so is Q.*

Note that it suffices to prove Proposition 16.11 for symmetric R because of the following observation. If R is invertible at X, then $T = R^{\mathrm{T}} R$ is also invertible at X and symmetric. Assuming the lemma for symmetric T, there is a Q so that $QT = 1$, but then $(QR^{\mathrm{T}})R = Q(R^{\mathrm{T}} R) = QT = 1$.

The proof of Proposition 16.11 for symmetric R relies upon the following lemma.

Lemma 16.12. *Suppose A, D are symmetric matrices and let $P(t) = A + 2tC^{\mathrm{T}}(I + tD)C$ for $t \in \mathbb{R}$. If for infinitely many $t \in \mathbb{R}$ there is a nonzero vector x_t satisfying $P(t)x_t = 0$, then there is a nonzero vector h satisfying $Ah = Ch = 0$.*

Proof. The hypothesis implies (e.g., by looking at $\det P(t)$) that for every $t \in \mathbb{R}$ there is a nonzero vector x_t satisfying $P(t)x_t = 0$. For $s \neq t$,

$$\langle P(t)x_t, x_s\rangle = 0 = \langle P(s)x_t, x_s\rangle$$

since $P(s)$ is symmetric. This gives,

$$0 = \langle (P(t) - P(s))x_t, x_s\rangle = (t - s)\langle 2C^{\mathrm{T}}(I + (t + s)D)Cx_t, x_s\rangle$$

and thus, for $t \neq s$,

$$0 = \langle 2C^{\mathrm{T}}(I + (t + s)D)Cx_t, x_s\rangle. \tag{16.1}$$

Choose a sequence s_j which strictly decreases to 0. Choose vectors x_{s_j} so that $\|x_{s_j}\| = 1$ and $P(s_j)x_{s_j} = 0$. There is a subsequence, still denoted s_j so that x_{s_j} converges to some y with $\|y\| = 1$. We have $0 = P(s_j)x_{s_j}$ converges to $P(0)y = 0$. Choosing $t = 0$ and $x_0 = y$ in equation (16.1) it follows that

$$0 = \langle 2C^{\mathrm{T}}(I + s_jD)Cy, x_{s_j}\rangle \to \langle 2C^{\mathrm{T}}Cy, y\rangle.$$

Thus $Cy = 0$. Since also $P(0)y = 0$, it follows that $Ay = 0$. $\square$

Proof of Proposition 16.11. The proof uses induction on m; suppose the proposition is true for $(m-1) \times (m-1)$ matrices.

Given an $m \times m$ matrix R of scalar rational expressions as in the hypothesis of the proposition, partition it as

$$R(x) = \begin{pmatrix} r_{11} & r_{12} \\ r_{12}^{\mathrm{T}} & r_{22} \end{pmatrix}. \tag{16.2}$$

Let

$$S = \begin{pmatrix} 1 & tr_{12} \\ 0 & 1 \end{pmatrix} \tag{16.3}$$

and consider the transformation $\tilde{R} = SRS^{\mathrm{T}}$ of R where $t \in \mathbb{R}$ is to be chosen shortly. Compute

$$\tilde{R} = \begin{pmatrix} r_{11} + 2t\, r_{12}r_{12}^{\mathrm{T}} + t^2 r_{12}r_{22}r_{12}^{\mathrm{T}} & r_{12} + t\, r_{12}r_{22} \\ r_{21} + t\, r_{22}r_{21} & r_{22} \end{pmatrix}.$$

Set $A := r_{11}(X)$ and $C := r_{12}(X)^{\mathrm{T}}$ and apply Lemma 16.12 to conclude that either A and C have a common null vector h or there exists a t so that the $(1,1)$ entry of $\tilde{R}(X)$ is invertible. In the first case,

$$R(X)\begin{pmatrix} h \\ 0 \end{pmatrix} = \begin{pmatrix} A & * \\ C & * \end{pmatrix}\begin{pmatrix} h \\ 0 \end{pmatrix} = 0.$$

which contradicts invertibility of $R(X)$.

Fix a t so that $r_{11} + 2t\ r_{12}r_{12}^{\mathrm{T}} + t^2 r_{12}r_{22}r_{12}^{\mathrm{T}}$ is invertible when evaluated at both X and 0. For notational ease, write

$$\tilde{R}(x) = \begin{pmatrix} a & b \\ b^{\mathrm{T}} & d \end{pmatrix}$$

for this choice of t. The condition that $a(0)$ is invertible means that a is invertible as a rational expression. Further, since both $a(0)$ and $\tilde{R}(0)$ are invertible, the Schur algorithm says that $(d - b^{\mathrm{T}}a^{-1}b)(0)$ is invertible. Similarly, $(d - b^{\mathrm{T}}a^{-1}b)(X)$ is invertible. Thus, the induction hypothesis says that the matrix $E = d - b^{\mathrm{T}}a^{-1}b$ of scalar rational expressions has an inverse; i.e., there is a matrix F of scalar rational expressions such that $FE = I$ and X is in the domain of F.

Now define a matrix rational expression Q by $Q = G^{\mathrm{T}}D^{-1}G$ where

$$G := \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad D := \begin{pmatrix} a & 0 \\ 0 & d - b^{\mathrm{T}}a^{-1}b \end{pmatrix}.$$

From above, the matrix D of scalar rational expressions has as an inverse which is a matrix of scalar rational expressions with X in the domain. Since the same is true for both G and S it follows that R has the desired inverse Q. □

A glance at the proof shows that we have established a somewhat stronger statement: given any finite (or countable) set of matrices $\{X_i\}$ in the domain of R so that $R(X_i)$ are invertible, there exists a matrix Q of rational expressions with $QR = I$ and X_i in the domain of Q for all i.

Proof of Theorem 16.10. The argument proceeds by induction. The point is that at each stage in the construction of a matrix rational expression R in $\mathfrak{R}$ which involves an inverse of some matrix rational expression, one can apply Proposition 16.11 to obtain that the induction step is valid. Since we have already done this type of induction twice (see §2.4 and Definition 16.1), we do not repeat it here. □

16.4.4. *Example Showing that Transformation (16.3) is Needed.* Let

$$R(x) = \begin{pmatrix} 1 - x_1 & x_2x_3 \\ x_3x_2 & 1 - x_4 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} \\ r_{12}^{\mathrm{T}} & r_{22} \end{pmatrix}$$

We shall choose an 2 tuple of symmetric matrices X at which $R(X)$ is invertible, but for any 2×2 matrices E, F $ERF(X)_{11}$ is never invertible.

Choose

$$X_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

With these choices,

$$R(X) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

In particular $R(X)$ is invertible. We search for matrices

$$E = \begin{pmatrix} a & b \\ * & * \end{pmatrix}, \quad F = \begin{pmatrix} c & * \\ d & * \end{pmatrix}$$

so that the $(1,1)$ entry of

$$(ERF)_{11} = ac\, r_{11} + bc\, r_{12}^{\mathrm{T}} + ad\, r_{12} + bd\, r_{22}$$

evaluated at X is invertible. However this is impossible, since

$$\begin{aligned} ERF(X)_{11} =& (ac\, r_{11} + bc\, r_{12}^{\mathrm{T}} + ad\, r_{12} + bd\, r_{22})(X) \\ =& \begin{pmatrix} ac & bc \\ da & db \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} a & c \end{pmatrix} \end{aligned}$$

is never invertible.

16.5. **Evaluating an NC Rational Function.** Evaluation of rational functions on g-tuples of matrices is central to this paper, so it has been discussed at the beginning of this paper, see 2.3. Notice that in the body of the paper we use everywhere only evaluation on g-tuples of symmetric matrices. Consequently we define symmetric domains $\mathcal{F}(n)_{r,\text{for}} = \mathcal{NF}(n)_{r,\text{for}} \cap (\mathbb{SR}^{n\times n})^g$. It follows from Proposition 16.7 item 5 that no information is lost by using symmetric evaluations only.

16.6. **General Noncommutative Rational Functions.** Although we do not need it in this paper, we mention that it is also possible to introduce noncommutative rational functions that are not necessarily analytic at the origin. We define noncommutative rational expressions (not necessarily analytic at the origin) as before, except that now we allow r^{-1} for any noncommutative rational expression r such that $\det r(X_1, \ldots, X_g)$ does not vanish identically for $(X_1, \ldots, X_g) \in \mathcal{NF}(n)_{r,\text{for}}$ for n large enough.

Let r be a noncommutative rational expression which is not equivalent to 0, i.e., such that $r(X_1, \ldots, X_g)$ does not vanish identically for $(X_1, \ldots, X_g) \in \mathcal{NF}(n)_{r,\text{for}}$ for n large enough (or, equivalently, for some n). Then necessarily $\det r(X_1, \ldots, X_g)$ does not vanish identically. (This follows since the algebra of generic matrices is embeddable in a skew field of fractions, namely its ring of central quotients, [Row80, Theorem 3.2.6].) Therefore noncommutative rational functions form a skew field of fractions of the ring of noncommutative polynomials. [5]

[5]Unlike in the commutative case, skew fields of fractions are not unique. The ring of noncommutative polynomials admits the so called universal skew field of fractions, see [Co71][Chapter 7] (notice in this connection that the ring of noncommutative polynomials is a fir). We conjecture that the universal skew field of fractions of the ring of noncommutative polynomials coincides with the skew field of noncommutative rational functions constructed here.

17. Appendix: Principal Ideals

This appendix deals with polynomials in commuting variables. Accordingly, let $\mathbb{R}[x_1, \ldots, x_n]$ **denote the polynomials in the commuting variables** $x_1, \ldots, x_n$. More generally we will use the notation/font x, t for commuting variables.

In our applications, the commuting variables arise naturally by fixing an m and considering the entries of the matrices in the g-tuple $X \in (\mathbb{SR}^{m\times m})^g$. Note, we have reserved the notation $\mathbb{R}\langle x_1, \ldots, x_g\rangle$ for polynomials in the non-commutative variables $\{x_1, \ldots, x_g\}$.

Also, in this appendix only we will violate our previous notation and use font x for points in $\mathbb{R}^n$.

Given a polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$, let $Z(p)$ denote the zero set of p,

$$Z(p) = \{x \in \mathbb{R}^n : p(x) = 0\}.$$

Proposition 17.1. *Suppose $r \in \mathbb{R}[x_1, \ldots, x_n]$ is irreducible, $\mathrm{x}^0 \in \mathrm{Z(r)}$, the gradient r' is not zero at x^0, and $\mathrm{x}^0 \in \mathrm{U} \subset \mathrm{Z(r)}$ is a $Z(r)$ relatively open set. If $q \in \mathbb{R}[x_1, \ldots, x_n]$ vanishes on U, then f divides q.*

Proposition 17.1 and its proof have been relegated to this appendix since it is likely not surprising to real algebraic geometers. Caution lead the authors to produce a proof in detail.

17.1. Proof of Proposition 17.1.

The proof relies heavily on Proposition 17.2 below found in [BCR98]. Recall, if $V \subset \mathbb{R}[x_1, \ldots, x_n]$ is a variety, then $I(V)$ denotes the ideal of the variety,

$$I(V) = \{f \in \mathbb{R}[x_1, \ldots, x_n] : f(\mathrm{x}) = 0 \text{ for all } \mathrm{x} \in \mathrm{V}\}.$$

If $I \subset \mathbb{R}[x_1, \ldots, x_n]$ is an ideal or even just a set then

$$V(I) = \{\mathrm{x} \in \mathbb{R}^n : f(\mathrm{x}) = 0 \text{ for all } f \in I\}$$

is the variety of the ideal. In particular, if $I = \{p\}$ is a singleton set, then $Z(r) = V(I)$. Finally, if $f_1, \ldots, f_\ell \in \mathbb{R}[x_1, \ldots, x_n]$,

$$I\langle f_1, \ldots, f_\ell\rangle = \{\sum_1^\ell g_j f_j : g_j \in \mathbb{R}[x_1, \ldots, x_n]\}$$

is the ideal generated by $\{f_1, \ldots, f_\ell\}$.

Proposition 17.2 (Theorem 4.5.1, page 94 [BCR98])**.** *If $f \in \mathbb{R}[x_1, \ldots, x_n]$ is irreducible and there is a point $\mathrm{y} \in \mathrm{R^n}$ such that $f(\mathrm{y}) = 0$ but $f'(\mathrm{y}) \neq 0$, then*

$$I(Z(f)) = I\langle f\rangle.$$

The algebraic dimension of a variety V may be defined as the maximal number of elements of $\mathbb{R}[V]$ which are algebraically independent over $\mathbb{R}$. Here are the relevant definitions.

Definition 17.3. The **coordinate ring** of V, denoted $\mathbb{R}[V]$, is the ring of polynomial functions on V. There is a natural identification,

$$R[V] = \mathbb{R}[x_1, \dots, x_n]/I(V).$$

Elements $f_1, \dots, f_\ell \in \mathbb{R}[x_1, \dots, x_n]$ are **algebraically dependent** (over $\mathbb{R}$), if there is a non zero $p \in \mathbb{R}[t_1, \dots, t_\ell]$ such that $p(f_1, \dots, f_\ell) = 0$.

Proposition 17.4. *The algebraic dimension of V is the maximal number of elements of $\mathbb{R}[V]$ which are algebraically independent.*

The dimension of V is also the largest d such that there exists $x_{j_1}, x_{j_2}, \dots, x_{j_d}$ such that

$$I(V) \cap \mathbb{R}[x_{j_1}, x_{j_2}, \dots, x_{j_d}] = \{0\}.$$

Of course, we have not defined the algebraic dimension of V, so we will take the first statement of the theorem as a definition and the second as a theorem. See [CLO92] Chapter 9.

Lemma 17.5. *Suppose $f \in \mathbb{R}[x_1, \dots, x_n]$ is irreducible (and not zero) and $I\langle f\rangle = I(Z(f))$. If $W \subset Z(f)$ is variety and if W has algebraic dimension $n-1$, then $W = Z(f)$. Explicitly, if a polynomial u is zero on W, then f divides u.*

Proof. The proof is modeled after the proof of Theorem 1, in §6, Chapter 1 of Shafarevich [Sh74]. Beware, in the edition [Sh74] there is some mix up in the varieties X and Y. Also beware that a standing hypothesis in this part of [Sh74] is that the base field is algebraically closed. Thus we follow a bit different trajectory.

Observe that the dimension of $Z(f)$ is at least as big as the dimension of W (obvious from Proposition 17.4). On the other hand, since f is not zero, the dimension of $Z(f)$ is at most $n-1$. Hence the algebraic dimension of $Z(f)$ is $n-1$.

Since the dimension of W is $n-1$, we can assume that $x_1, \dots, x_{n-1}$ are algebraically independent on W. Hence they are algebraically independent on $Z(f)$. Fix $u \in \mathbb{R}[Z(f)]$ and assume $u \neq 0$. Since the dimension of W is strictly smaller than n, it follows that the set $\{x_1, \dots, x_{n-1}, u\}$ is algebraically dependent. Consequently, there exists an ℓ and $a_0, \dots, a_\ell \in \mathbb{R}[x_1, \dots, x_{n-1}]$ satisfying

$$a_\ell u^\ell + a_{\ell-1}u^{\ell-1} + \dots + a_1 u + a_0 = 0 \quad \text{on } Z(f). \tag{17.1}$$

Assume ℓ is the smallest integer for which there exists $a_0, a_1, \dots, a_\ell$ such that equation (17.1) holds on $Z(f)$. Note that equation (17.1) automatically holds on the smaller set W.

The fact that a_0 does not vanish on $Z(f)$ uses the hypothesis that f is irreducible. By way of contradiction, suppose a_0 does vanish on $Z(f)$. Then, $ug = 0$ on $Z(f)$ where

$$g = a_\ell u^{\ell-1} + \dots + a_2 u + a_1.$$

Since, by hypothesis, $I(Z(f)) = I\langle f\rangle$ and $ug = 0$ on $Z(f)$, there is an h such that $hf = ug$. Thus f divides ug and since f is irreducible, f divides either u or g. If f divides u, then $u = 0$ on $Z(f)$, contrary to hypothesis. On the other hand, if f divides g, then $g = 0$ on $Z(f)$, contradicting the minimality of ℓ. We conclude that a_0 is not zero on $Z(f)$.

Now suppose $u = 0$ on W. It follows from equation (17.1) that $a_0 = 0$ on W. Since $x_1, \dots, x_{n-1}$ are algebraically independent on W, it follows that $a_0 = 0$ contradicting a_0 is not zero on $Z(f)$. Thus, u is not zero on W.

We have shown, if $u \in \mathbb{R}[Z(f)]$ is zero on W, then u is zero on $Z(f)$. Hence, $W = Z(f)$. Further, as u is zero on $Z(f)$ and $I(Z(f)) = I\langle f\rangle$, we obtain f divides u. $\square$

Lemma 17.6. *Suppose $f \in \mathbb{R}[x_1, \dots, x_n]$ is irreducible (and not zero),* $\mathrm{x}^0 \in \mathrm{Z(f)}$*,* $f'(\mathrm{x}^0) \neq 0$*, and* $\mathrm{x}^0 \in \mathrm{U} \subset \mathrm{Z(f)}$ *is open relative to $Z(f)$. If $q \in \mathbb{R}[x_1, \dots, x_n]$ vanishes on U, then q vanishes on $Z(f)$ and thus f divides q.*

Proof. The proof is modeled after the proof of Theorem 2 of Chapter 1 §6 in [Sh74]. Again, beware that theorem is for an algebraically closed field. We have other hypotheses to compensate.

Let $W = Z(q^2 + f^2)$. Then $W \subset Z(f)$ is a variety and so most of the proof consists of showing that W has algebraic dimension $n - 1$, so that we may apply Lemma 17.5. We are assuming that $f'(\mathrm{x}^0) \neq 0$, so we may assume that

$$\frac{\partial f}{\partial x_n}(\mathrm{x}^0) \neq 0.$$

Suppose there is a polynomial G in $t_1, \dots, t_{n-1}$ such that $G(t_1, \dots, t_{n-1})$ is zero on W. The hypothesis $f'(\mathrm{x}^0) \neq 0$ means that we may apply the implicit function theorem and conclude that there is a neighborhood U^0 of x^0 in $Z(f)$ and an open set $N \subset \mathbb{R}^{n-1}$ such that

$$N = \pi(Z(f) \cap U^0),$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection onto the first $n-1$ coordinates. Thus, G vanishes on N and since N is open, G is identically zero. Hence the dimension of W is at least $n - 1$.

An application of Lemma 17.5 now says that $W = Z(f)$ from which it follows that $Z(q) \supset Z(f)$. Thus, f divides q. $\square$

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