October 1, 2003

Title "Operator Theory Arising from Systems Engineering"

Contents

A. PROJECT SUMMARY

The classical core

This is a proposal to continue support for several projects in the parts of operator theory and functional analysis related to engineering system theory. One branch of analysis closely related to applications is classical Nevanlinna-Pick-Nehari theory, or equivalently commutant lifting theory, a part of the area called operator model theory. The early development of this was done for the purest of mathematical reasons, but in the mid 1970's and early 1980's this was shown to be critical to the design of engineering systems where stability of the system is the key constraint. This motivated much more mathematical development and now it is one of the areas of functional analysis most closely associated with control engineering. For many years (since Norbert Wiener) engineering design tools optimized mean square performance. The theory above ultimately lead to (commercially commonplace) tools for optimizing worst case frequency domain performance. The main development in linear systems engineering in the 1990s was the development of powerful matrix inequality techniques which substantially generalize these core results. Convexity is particularly desirable for the numerics they use. *The goal of much of this research is to extend this theory in radically new directions.*

1. Noncommutative inequalities: Matrix-positive polynomials Abbreviate noncommutative by NC. We consider NC polynomials, $p(x, x^T)$, in variables $x = \{x_1, \cdots x_g\}$ etc. which respect an involution denoted ^{T}. We call a symmetric NC polynomial **matrix-positive** provided that when we substitute into p any matrices $X_1, \cdots, X_g$ of any dimension $\eta \times \eta$ for $x_1, \cdots, x_g$, and their transposes, $X_1^T, \cdots, X_g^T$ for $x_1^T, \cdots, x_g^T$, the resulting matrix $p(X_1, \cdots, X_g, X_1^T, \cdots, X_g^T)$ is positive semi-de£nite. **Matrix convexity** is de£ned analogously. A polynomial p is a **Sum of Squares, SoS,** provided $p(x, x^T) = \sum_{i=1}^k h_i(x, x^T)^T h_i(x, x^T)$ for some NC polynomials h_i.

thm:PosIntro **Theorem 0.1 (H01)** *Every matrix positive symmetric noncommutative polynomial can be written as a SoS.*

It was known to Hilbert that positive commutative polynomials may not be a sum of squares of polynomials (eg. his 17^{th} problem). More generally, the classical area of semialgebraic geometry (a branch of real algebraic geometry) gives a systematic theory of polynomial inequalities. McCullough, Putinar and I are successfully extending semialgebraic geometry to NC polynomials and to NC rational functions where we analyse their action on variables which are operators on a Hilbert space $\mathcal{H}$. This has elegant behavior and seems like a rich new area.

2. Matrix Convexity: Theorem 0.1 plus matrix convexity being equivalent to matrix positivity of a NC 2^{nd} derivative gives:

thm:convp1 **Theorem 0.2 (MHtrans)** *Every matrix convex symmetric NC polynomial has degree two or less.*

3. Convexity vs LMIs We say a set $\mathcal{C} \subset \mathbf{R}^g$ has a **Linear Matrix Inequality (LMI) Representation** provided that there are real entried symmetric matrices $L_0, L_1, L_2, \cdots, L_g$ for which the set

$$\{s = (s_1, s_2, \cdots s_g) \in \mathbf{R}^g : \ L_0 + L_1 s_1 + \cdots + L_g s_g \quad \text{is PosSemiDef}\}$$

equals the set $\mathcal{C}$. Similarly in **Noncommutative LMI Representations (NCLMI)** we take the s_j to be operators on a Hilbert space $\mathcal{H}$. V. Vinnikov and Helton [HVprept] *described precisely which convex sets in* $\mathbf{R}^g$, $g = 2$ *have an LMI representation.* The characterization is elegant and A. S. Lewis, P. A. Parrilo and M. V. Ramana [arXive 2003] used our result to verify a 1958 conjecture of Peter Lax as an immediate consequence. This was one of two results featured by the IMA (Minnesota) in its summer 2003 report. Currently we are considering the $g > 2$ case; it seems within reach.

We have results, stronger than Theorem 0.2 suggesting that NC convexity and LMIs have a remarkably close relationship, including much progress on our conjecture a rather special case of which is

Conjecture *Let p be a symmetric NC polynomial. The positivity set* $\mathcal{D}_p := \{X = \{X_1, \cdots, X_g\} : \ X_j \ an \ operator \ on \ \mathcal{H} \ making \ p(X) \ PosDef\}$ *of p is convex iff* $\mathcal{D}_p$ *has a "noncommutative LMI" representation.*
Topics 1, 2, and 3 were subjects of plenary talks at SIAM Control Conf 2001 and the MTNS 2002.

4. Optimization over spaces of analytic functions and matrices (with Camino, H. Dym, and Skelton). We do: Qualitative theory, computer algorithms based on this theory, analysis of such algorithms, connections with other branches of mathematics. These are key optimization problems arising in designs of linear systems.

5. Highly nonlinear generalizations. The goal is to £nd canonical "nonlinear generalizations" of analytic function theory and the related parts of operator theory. There are many strong results and currently we are trying to extend linear matrix inequality results to nonlinear operators.

6. Computer operator algebra. Linear engineering systems theory and operator theory are rife with calculations in a noncommutative algebra. Helton's group and M. Stankus are the main providers of software (called NCAlgebra) for performing general noncommuting calculations in Mathematica. An emphasis now is on algorithms for treating NC inequalities. Also we experiment on many problems with NC Gröbner bases a powerful technique.

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FASTLANE WILL PROVIDE

C. PROJECT DESCRIPTION

RESULTS FROM PRIOR SUPPORT

NSF Award Number DMS 9732891 From 7/1/98 to 6/30/01 for $150,000
NSF Award Number DMS 0100576 From 7/1/01 to 6/30/04 for $239,072.00

ARTICLES and PREPRINTS for the last £ve years of NSF support (from 1998 on were):

[HSW98], [YJH98], [HMW98], [AHS98], [HMer98], [HKD98], [HMer98], [HJ98], [HJM98], [HS99], [H99], [HD99], [HJ99], [DHM99], [HD99], [HDJ99], [HHK99], [HK99], [HHK00], [HKMS00], [CHS00], [HW00], [H01], [SHAPC01], [HJM01], [SHAPCb01], [H02], [DHM02], [H02] , [DHM03], [CHSY03], [OH03], [OH03cdc], [HWsicon], [HM trans], [HMsimax], [HMPcre], [HSro], [HJMprept1], [HJMprept2], [HJMprept3], [HVprept], [HMPprept], [BHRprept], [HMerprept], [HHKprept], [HHKprept],

A cohesive account of how this work relates to the proposed work appears later.

IMPACT ON HUMAN RESOURCES

My computational efforts have generated a small 'lab' in the summer (the lab exists because of a core of NSF funds). Students learn some math, develop computing skill and experience with math experiments, and get a taste of systems engineering. Also they produced some of the software and results described in this proposal. In the lab over 5 years, there were 12 graduate students and 2 undergrads. Some of these people spent several summers on the project. Also partially supported from my grants were 4 postdocs. See also the students in the Synergistic Activities section.

My work helps signi£cant collections of mathematicians in the US to have an awareness of engineering problems and how mathematics techniques connect with them. See the biographical section, Synergistic Activities.

RESEARCH PLAN

1 OPERATOR THEORY & NONCOMMUTATIVE INEQUALITIES

sec:linOp

Inequalities involving polynomials in matrices and their inverses and associated optimization problems have become very important in control and systems engineering; this was the main development in linear systems theory in the 1990's. When these polynomials are linear, the resulting matrix inequalities are called **Linear Matrix Inequalities, LMI,** and various numerical algorithms such as interior point methods apply directly. A dif£culty is that often an engineering problem presents a matrix rational function problem whose conversion to an LMI takes considerable skill, time, and luck to determine. Typically this is done by looking at a formula and recognizing complicated patterns involving Schur complements; a tricky hit or miss procedure; § sec:motiv 5 discusses such motivation. The goal here is to begin to develop a systematic mathematical theory which pertains to these areas. This leads us directly into mathematics problems very well suited for study by operator theorists.

We start discussion of matrix inequalities not by directly discussing "Linear Noncommutative Inequalities", or "Noncommutative Convexity", but by describing some results on positive polynomials which feed into this work. This amounts to a natural version of the classical subject of (real) semialgebraic geometry for noncommutative polynomials.

1.1 Matrix Positive Polynomials

sec:intro

A classic problem which goes back to Hilbert asks which real polynomials are a sum of squares. It is well known that many positive polynomials are not a sum of squares. We consider a class of noncommutative polynomials and show that if they are positive in a certain sense, then they are a sum of squares.

Abbreviate noncommutative by NC. We consider polynomials, $p(x, x^T)$, in noncommutative indeterminants $x = \{x_1, \cdots x_g\}$ and $x^T = \{x_1^T, \cdots x_g^T\}$ which respect an involution denoted ^{T}, denote these by $\mathcal{N}_*$. Call a polynomial p symmetric if $p^T = p$, and we call one **matrix-positive** provided that when we substitute into p any matrices $X_1, \cdots, X_g$ of any dimension $\eta \times \eta$ for $X_1, \cdots, X_g$, and their transposes, $X_1^T, \cdots, X_g^T$ for $x_1^T, \cdots, x_g^T$, the resulting matrix $p(X_1, \cdots, X_g, X_1^T, \cdots, X_g^T)$ is positive semi-de£nite.

Example. Let $p \in \mathcal{N}_*$ be given by $p = x_3x_2 + 3x_3x_1x_2$ and note $p^T = x_2x_3 + 3x_2x_1x_3$. If $X = (X_1, X_2, X_3)$, where

$$X_1 = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ then\ p(X) = \begin{pmatrix} 0 & 0 \\ -5 & 3 \end{pmatrix} \text{ and } p^T(X) = \begin{pmatrix} 0 & -5 \\ 0 & 3 \end{pmatrix}.$$

If $q \in \mathcal{N}_*$ is the symmetric polynomial $q = p + p^T$, then $q(X) = \begin{pmatrix} 0 & -5 \\ -5 & 6 \end{pmatrix}$. ■

Throughout lower case letters like x_j will stand for variables (indeterminants) while corresponding capital letters like X_j will stand for operators on a Hilbert space $\mathcal{H}$ substituted for them. Often $\mathcal{H}$ will be £nite dimensional in which case we call X_j matrices. We say a polynomial p is a **Sum of Squares, SoS,** provided there are NC polynomials h_i such that

$$p(x, x^T) = \sum_{i=1}^{k} h_i(x, x^T)^T h_i(x, x^T) \tag{1}$$

eqn:SoS

Theorem 1.1 *Every matrix positive symmetric noncommutative polynomial is a sum of squares*

It was known to Hilbert that positive polynomials may not be a sum of squares of polynomials (eg. his 17^{th} problem). The area of real semialgebraic geometry concerns this and more generally gives a systematic theory of polynomial inequalities. A theorem at the center of that subject is the positivestellensatz which we describe next.

1.2 A Strict NC Positivstellensatz

sec:posSS

Let $\mathcal{P}$ denote a collection of symmetric polynomials in noncommutative variables $x = \{x_1, \cdots, x_g\}$. The **positivity domain** $\mathcal{D}_\mathcal{P}$ **associated to** $\mathcal{P}$ is the set of tuples $X = (X_1, \cdots, X_g)$ of symmetric bounded operators on separable real Hilbert space $\mathcal{H}$ making $p(X_1, \cdots, X_g)$ a positive semi-de£nite operator for each $p \in \mathcal{P}$. The domain is bounded by κ if there exists a scalar κ so that $\kappa^2 I - X_j^T X_j$ is positive semi-de£nite whenever $X \in \mathcal{D}_\mathcal{P}$.

Theorem 1.2 (HMtrans) **Strict Positivstellensatz, PosSS** *Suppose $\mathcal{D}_\mathcal{P}$ is bounded by κ. A NC symmetric polynomial q which is strictly positive on $\mathcal{D}_\mathcal{P}$ can be written as a weighted sum of squares*

$$q = \sum_{1}^{N} s_j^T p_j s_j + \sum_{1}^{M} r_k^T r_k + \sum_{\ell=1}^{g} \sum_{m=1}^{N_\ell} t_{m,\ell}^T w_\ell t_{m,\ell} \tag{2}$$

eq:intro

for polynomials $p_j \in \mathcal{P}$, $w_\ell = \kappa^2 - x_\ell^T x_\ell$ and some NC polynomials $s_j, r_k, t_{m,\ell}$.

When $\mathcal{D}_\mathcal{P}$ is a convex set, the Hilbert space $\mathcal{H}$ used in our positivity hypothesis can be taken to be £nite dimensional. Versions of this positivstellensatz are also proved for two additional classes of matrix-valued noncommutative polynomials.

Pablo Parrilo developed a very effective algorithm for £nding commutative SoS and PosSS representations numerically. It is based on ideas for algorithms by N. Z. Shor, Powers-Wormann and Reznick. Pablo Parrilo observed that the critical problem here is an LMI. Software for solving LMIs abounds, see *http://plato.la.asu.edu/dimacs.html* for results of benchmark comparisons of numerical packages for solving LMIs. The algorithm also works to implement the representation in the positivstellensatz for both commutative and noncommutative polynomials. The commutative version, *SOS Tools*, is available at *http://control.ee.ethz.ch/ p̃arrilo/sostools.* Parrilo and collaborators have many applications of the commutative version of this theorem, eg. to quantum entanglement, £nding Lyapunov functions, and various graph theoretic problems.

1.3 Noncommutative Real Nullstellensatz

sec:nullSS

What happens when $q(X)$ is positive semide£nite but not positive de£nite for $X \in \mathcal{D}_\mathcal{P}$? For commutative polynomials an extreme case is the real nullstellensatz concerning zeroes of polynomials. Now we turn to the noncommutative situation. Suppose p is a NC polynomial. De£ne its zero set

$$Z_p := \{(X, v) :\ X = (X_1, \ldots, X_g)\ a\ tuple\ of\ operators\ acting\ on\ the\ Hilbert\ space\ \mathcal{H}$$
$$and\ a\ vector\ v \in \mathcal{H}\ for\ which\ p(X)v = 0\}.$$

1.3.1 Polynomials in x and x^T: Bad news

One conceivable weak version of a noncommutative real nullstellensatz we investigated goes like this: Suppose p and q symmetric NC polynomials from $\mathcal{N}_*$. If $Z_p \subset Z_q$, then does it follow that

$$q^{2m} + R = pr + r^T p \tag{3}$$

eq:rnull

for some positive integer m, polynomial $r \in \mathcal{N}_*$, and R a sum of squares, $R = \sum r_j^T r_j$ for $r_j \in \mathcal{N}_*$?

The example in [HMtrans],

$$p = (x^T x + x x^T)^2 \quad and \quad q = x + x^T, \tag{4}$$

eq:exN

where x is a single variable is shown to prove the representation in (3) is *false*.

eq:rnull

1.3.2 The Transpose Free Nullstellensatz: Good news

subsec:nullstatz

The case where p and q are polynomials purely in x and the "matrix zero set" of q contains that of p yields a satisfying result (conjectured by McCullough and Helton) proved by George Bergman, see [HMtrans].

Theorem 1.3 *Fix a £nite collection $\mathcal{P}$ of polynomials in noncommuting variables $\{x_1, \ldots, x_g\}$ and let q be a given polynomial in $\{x_1, \ldots, x_g\}$. Let d denote the maximum of the deg(q) and $\{deg(p) : p \in \mathcal{P}\}$. Let $\mathcal{H}$ be a real Hilbert space of dimension $\sum_{j=0}^{d} g^j$. If $Z_p \subset Z_q$ for all $p \in \mathcal{P}$, then q is in the left ideal generated by $\mathcal{P}$.*

Question: *If p contains no transposes, and q is symmetric, then does the representation (3) hold?* eq:rnull
This and many variations of it are under study at the moment.

1.4 Non Strict Positivstellensatz

subsec:inbtw

The example in equation (4) eq:exN gives strong evidence that a nice characterization of the "radical ideal" is unlikely. However, in other respects the nonstrict case may be well behaved.
Question: *Suppose q is a symmetric NC polynomial. If $q(X)$ is PosSemiDef on $\{X : \ p(X) = 0\}$, then is there a SoS, $R := \sum_j r_j^T r_j$, such that*

$$q(X) = R(X) \text{ for all } X \text{ satisfying } p(X) = 0?$$

thm:sphere

Theorem 1.4 *[HMPcre] For $p = 1 - \sum_j^g x_j^T x_j$, the answer is yes.This makes no mention of characterizing the radical ideal of p and is false in the commutative case. Indeed, the answer in the general NC case is still an intriguing mystery.*

1.5 A Matrix Based Positivstellensatz

subsec:posMB

Consider a noncommutative rational function $q(x)[h]$ in two different types of variables: distinguished by the fact that q is quadratic in h. Represent $q(x)[h]$ as

$$q(x)[h] = V(x)[h]^T M_q(x) V(x)[h], \tag{5}$$

where $M_q(x)$ is a symmetric matrix with rational entries, $V(x)[h]$ is linear in h, rational in x, and the h variables sit to the left of all x variables. Now we roughly summarize our main result on this type of representation.

EX:NC

Result 1.1 *(see Theorem 8.2 of [CHSY03]).* **Positivity:** q ***versus*** M_q**.**
Let $\mathcal{G}$ denote the positivity domain based on $M_q(x)$, namely,

$$\mathcal{G} := \{X : \ M_q(X) \ is \ PosSemiDef\}.$$

Then $q(X)[H]$ is PosSemiDef for each $X \in \mathcal{G}$ and for all H.
Conversely, assume:

1. *all h_j are restricted to be symmetric (other cases also work well);*
2. *the M_q representation of q has a $V[h]$ meeting a simple linear independence condition;*
3. *the inequality domain $\mathcal{G}$ contains an open set in a certain topology and is the closure of that set.*

Then the closure of $\mathcal{G}$ in that same topology is the biggest domain on which $q(X)[H]$ is a matrix positive quadratic.

This is something of a Positivestellensatz for noncommutative quadratics q, a very special class of functions, but the conclusions are more re£ned in that they give precisely the "set of positivity" for q. Also strictness is not required. This is very useful in convexity work to soon be described.

To further analyze positivity of q we apply the noncommutative LDL decomposition to the matrix $M_q(x)$, i.e., $M_q(x) = LDL^T$. The diagonal matrix D has the form

$$D = diag\,\{\rho_1(x), \cdots, \rho_c(x)\}$$

We use this to show that under reasonable hypotheses, q *has a weighted sum of squares decomposition*

$$q(x)[h] = \sum_{j=1}^{r} \mathcal{L}_j(x)[h]^T D_j(x) \mathcal{L}_j(x)[h]$$

with $\mathcal{L}_j, D_j$ rational and $\mathcal{L}_j$ linear in h, such that formal inequalities involving the D_j determine a set

$$\mathcal{G} := \{X : D_j(X) \; is \; PosDef\}$$

Moreover, a certain "closure" of $\mathcal{G}$ is the largest such set.

McCullough, Putinar and I have extended Result 1.1 to polynomials in two classes of variables x, h which are *not necessarily quadratic* in h, [HMPcre].

Open questions: *Extending Result 1.1 to several simultaneous matrix inequalities is important. This corresponds to having several noncommutative rational functions. Replacing invertibility requirements on the $D_j(X)$ by pseudo-invertibility is an issue which is uncharted (and not too high on my list).*

sec:lin.1.2

2 MATRIX CONVEX FUNCTIONS

2.1 De£nition of Matrix Convexity

When A and B are symmetric matrices, the notation

$$A > B \; means \; A - B \; is \; PosDef \quad and \quad A \geq B \; means \; A - B \; is \; PosSemiDef.$$

A noncommutative rational symmetric function Γ of $x = \{x_1, \ldots, x_g\}$ will be called **geometrically matrix convex** provided that whenever the noncommutative variables x are taken to be any matrices$X$, then for all scalars $0 \leq \alpha \leq 1$ we have that

$$\alpha\Gamma(X^1) + (1-\alpha)\Gamma(X^2) \geq \Gamma(\alpha X^1 + (1-\alpha)X^2)$$

Here $X^1 = \{X_1^1, \ldots, X_g^1\}$ and $X^2 = \{X_1^2, \ldots, X_g^2\}$ are g-tuples of matrices. We are all familiar with the fact that for an ordinary (commutative) function f, convexity corresponds to positivity of the second derivatives of f. Now we describe a quite practical noncommutative version of this.

Noncommutative Functions and their Derivatives Noncommutative rational functions of indeterminants a, x, y, etc are polynomials in a, x, y, etc and in inverses of polynomials in a, x, y, etc. For example, F in (9) can be regarded as a noncommutative rational function of indeterminants, rather than a function on matrices. A notion of second derivative of a function Γ which we £nd useful for noncommutative symbolic computation is that of directional second derivative: namely, the second order terms (the Hessian) of a Taylor expansion of $\Gamma(x + th)$ about $t = 0 \in \mathbf{R}$ in the direction h. Thus we de£ne the **Hessian** of Γ, denoted by $Hess\Gamma$, as

$$Hess\Gamma(x)[h] := \frac{d^2}{dt^2}\Gamma(a, x+th)\Big|_{t=0}.$$

This is a quadratic function of $h := \{h_1, \ldots, h_g\}$. The function $\Gamma(x)$ is said to be **matrix convex** with respect to variable x on a positivity domain $\mathcal{D}$ provided its Hessian $Hess\Gamma(X)[H]$ is a positive semide£nite matrix for all A, X in $\mathcal{D}$ and all H.

One can show that matrix convexity is equivalent to matrix positivity of the noncommutative second derivative, see [HMercdc98]. Our positivity theory described above applies to this second derivative and leads to strong results on convexity.

2.2 NC Polynomials

thm:ConvPoly

Theorem 2.1 *(MHtrans) Every symmetric NC polynomial which is matrix convex (even if just "near $x = 0$") has degree two or less.*

The one variable case has been known for 20 years and is due to T. Ando. There is no parallel to this very rigid behavior in the commutative case.

sec:ConvRat

2.3 NC Rational Functions and Linear Pencils

The issue at hand is how does the extremely rigid structure imposed on NC polynomials by convexity extend to NC rational functions. We believe that it is closely allied to "linear pencils", the key ingredient of LMIs, so popular these days in engineering .

We shall use the term **linear pencil** to refer to a family of symmetric matrices

$$L(x) := L_0 + L_1 s_1 + \cdots + L_g s_g$$

where $s = (s_1, \cdots, s_g)$ are g real scalars and $L_0, L_1, L_2, \cdots L_g$ are symmetric real $d \times d$ matrices, and we call the pencil **monic** if $L_0 = I$. A **NC linear pencil** is

$$L(x) := L_0 \otimes I + L_1 \otimes x_1 + L_2 \otimes x_2 + \cdots + L_g \otimes x_g \tag{6}$$ eq:NCpen

in symmetric noncommuting variables x_j. Then, if X is a tuple of matrices or operators, $L(X)$ is formed from (6) eq:NCpen using terms $L_j \otimes X_j$ where $\otimes$ denotes tensor product. We now give an example.

Example Take $g = 2, \mathrm{d} = 2$ and $L_0 := I$ and

$$L_1 := \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}, \qquad L_2 := \begin{pmatrix} 3 & 5 \\ 5 & 0 \end{pmatrix}, \quad then \;\; L_1 \otimes X_1 := \begin{pmatrix} 2X_1 & 3X_1 \\ 3X_1 & 0 \end{pmatrix}$$

and the pencil $L(X)$ is

$$L(X) = \begin{pmatrix} 1 + 2X_1 + 3X_2 & 3X_1 + 5X_2 \\ 3X_1 + 5X_2 & 1 \end{pmatrix}. \tag{7}$$ eq:Ldef1

Clearly, if we were to replace X by 2-tuples of 3×3 matrices, then $L(X)$ would be a 6×6 matrix.

Conjecture *A large class $\mathcal{C}$ of symmetric NC rational functions has the property that $r \in \mathcal{C}$ is matrix convex "near x=0" if and only if it has the form*

$$r(x) = C[L_0 \otimes I + L_1 \otimes x_1 + L_2 \otimes x_2 + \cdots + L_g \otimes x_g]^{-1} \, [L_2 \otimes x_2 + \cdots + L_g \otimes x_g] C^T + D. \tag{8}$$ eq:repTF

for some $D \in \mathbf{R}^1$, $C \in \mathbf{R}^{1 \times d}$ and some symmetric $L_j \in \mathbf{R}^{d \times d}$, having $L_0 = I$. Moreover, the set of X with X_j operators on $\mathcal{H}$ which make $r(X)$ £nite has connected component of 0 denoted by $\mathcal{F}$ and r on $\mathcal{F}$ is matrix convex.

There is considerable progress by Helton, McCullough and Vinnikov; and we think it is likely that our layout of a proof will work. Critical are the realization results in progress by Joe Ball and T. Malakorn, extending those of M. Fliess from 25 years ago. Our proofs suggest the natural class $\mathcal{C}$ of NC rational functions for the Conjecture may be those satisfying a growth estimate at in£nity. Currently algorithms for computing this type of realization are under investigation for implementation in NCAlgebra, see § sec:compMisc 6.3.

This problem bears directly on the NC LMI representation problem to be discussed in § sec:LMIrep 3.2.

2.4 Convexity Checking Algorithm

sec:ConvCheck

The positivstellensatz theory then leads to and validates a symbolic algorithm for determining regions of convexity which is currently implemented in NCAlgebra. We introduce the topic with an example of an NCAlgebra command (which embodies it). Our command is

NCConvexityRegion[Function Γ, x].

It actually runs at greater generality than our de£nitions allow, since to save space in this proposal we have worked at a low level of generality. When we input a noncommutative rational function $\Gamma(a, x)$ this command outputs a family of inequalities which determine a domain $\mathcal{G}$ of a, x on which Γ is "matrix convex" in x. This is illustrated by

Example Suppose one wished to determine the domain of convexity (or concavity) of the following function on matrices, where $x = x^T, y = y^T$:

$$F(x, y) := -(y + a^T x b)(r + b^T x b)^{-1}(y + b^T x a) + a^T x a. \tag{9}$$ ex:5.2b

Apply the command **NCConvexityRegion**[F, $\{x, y\}$]. This command outputs the list

$$\{-2\,(r + b^T x b)^{-1},\; 0,\; 0,\; 0\}.$$

This output has the meaning that whenever A, B, R are £xed matrices, the function F is "x, y-matrix concave" on the domain of matrices X, and Y

$$\mathcal{G}_{A,B,R} := \{(X, Y) : (R + B^T X B)^{-1} > 0\}.$$

The command NCConvexityRegion also has an important feature which for this problem assures us no domain bigger than

$$\bar{\mathcal{G}}_{A,B,R} := \{(X, Y) : R + B^T X B \geq 0\}$$

is a "domain of concavity" for F. The last assertion requires the theory and proofs in § subsec:posMB 1.5.

S:A **Our Convexity Algorithm**

1. Compute symbolically the Hessian $q(x)[h] := Hess\Gamma(x)\ [h]$, represent $q(x)[h]$ as $q(x)[h] = V(a,x)[h]^T M_q(a,x) V(a,x)[h]$, and extract the coefficient matrix $M_q(a,x)$.

2. Apply the noncommutative LDL decomposition to the matrix $M_q(a,x)$, i.e., $M_q(a,x) = LDL^T$. The diagonal matrix D has the form $D = diag\ \{\rho_1(a,x), \dots, \rho_c(a,x)\}$

3. The Hessian $q(X)[H]$ is positive semidefinite for all H on the set of matrix tuples A, X which makes the diagonal matrix D positive semidefinite. Thus a set $\mathcal{D}$ where Γ is matrix convex is given by

$$\mathcal{D} = \{A, X : \rho_j(A,X)\ \ is\ PosDef,\ j = 1, \dots, c\}. \quad (10)$$

eq:algGdef

The surprising and deep feature is that the closure of $\mathcal{D}$ is the largest possible domain of convexity. This requires the NC Positivity Theory of § 1.5. Also it is hard to imagine precise "convexity region algorithm" not based on noncommutative calculations, the problem being that matrices of practical size often have thousands of entries, so would lead to calculations with huge numbers of polynomials in thousands of variables.

The algorithm and Mathematica implementation of NCConvexityRegion[ ] was done jointly with graduate students Juan Camino and Josh Griffin. Experiments with this software suggested Theorem 2.1 and the conjectures in § 2.3 as well as valid directions for proof. Algorithms which improve our NC LDU decomposition, eg. simplifying the pivots (rational functions) is a big issue, are helpful for upgrading NCConvexityRegion[ ]. Algorithms in new directions will be mentioned later.

3 CONVEX SETS vs. LMI REPRESENTATIONS

sec:LMIrepCom

3.1 Commutative LMIs

We say a set $\mathcal{C} \subset \mathbf{R}^g$ has a **Linear Matrix Inequality Representation** provided that there are real symmetric matrices $L_0, L_1, L_2, \cdots, L_g$ for which the set

$$\mathcal{D}_L := \{s = (s_1, s_2, \cdots s_g) \in \mathbf{R}^g :\ L_0 + L_1 s_1 + \cdots + L_g s_g \quad \text{is PosSemiDef}\}$$

equals the set $\mathcal{C}$. That is, $\mathcal{C}$ is the positivity set of some linear pencil. Obviously, an LMI representable set must be convex. So the issue is how close is the converse to being true. Without loss it is possible to take $L_0 = I$ and take $0 \in \mathcal{C}$.

In the many applications which LMIs have found, there is no systematic way to produce LMIs for general classes of problems. Before there is hope of producing LMIs systematically one must have a good idea of which types of constraint sets convert to LMIs and which do not. This seems like a fundamental issue regarding LMI's and was stated formally as an open question for $g = 2$ by Pablo Parrilo and Berndt Sturmfels in a 2001 preprint. We shall sketch briefly a solution developed with Victor Vinnikov in [HVprept] .

We say a polynomial on $\mathbf{R}^g$ satisfies the **real zeroes condition RZ** if for every $s \in \mathbf{R}^g,\ s \neq 0$, all the zeroes of $p(ts)$ for t complex are real numbers. Call such p an **RZ polynomial**.

thm:lmirep

Theorem 3.1 (HVprept) *Assume $g = 2$. An RZ polynomial p is the determinant of some monic linear pencil L, that is $p(s) = detL(s)$. A convex set $\mathcal{C}$ containing 0 has an LMI representation iff the minimal degree polynomial whose zero set contains the boundary of $\mathcal{C}$ is an RZ polynomial.*

The proof is not at all elementary and involves a considerable amount of Riemann surface theory. The proof is based on earlier work of V. Vinnikov [V89], [V93] and improved by more recent work with J. A. Ball [BV96] [BV99]. The proof itself can be found in [HVprept]. There we give two proofs. One of them is (in principle) a self contained construction of the L_0, L_1, L_2 based on Riemann surface theory, θ functions and such.

A. S. Lewis, P. A. Parrilo and M. V. Ramana [prept 2003] used this result to immediately solve a 1958 conjecture of Peter Lax to the effect that "hyperbolic" polynomials have such pencil representations. A hyperbolic polynomial is a type of homogenization of an RZ polynomial. Their result was the first of two pieces of research featured for Summer 2003 by the Minnesota IMA, see *http://www.ima.umn.edu/newsltrs/updates/summer03/outcomes.shtml*

Currently Vinnikov and I have two separate lines of argument for the $g > 2$ cases which leads us to

Conjecture *Theorem 3.1 including the 1958 Lax conjecture is true for all dimensions g.*

Question (raised by Nestrov and Nimmerovski in 1994) *Find a test which insures that a convex set $\mathcal{C}$ in $\mathbb{R}^g$ lifts to some LMI representable set $\tilde{C}$ in a possibly bigger space $\mathbb{R}^{g+k}$. That is, find necessary and sufficient properties on a given set $\mathcal{C}$ in $\mathbf{R}^g$ which insure that there exists an LMI representable set $\tilde{C}$ in $\mathbb{R}^{g+k}$ whose projection onto $\mathbb{R}^g$ equals the set $\mathcal{C}$?*

No serious apriori restrictions on such a convex $\mathcal{C}$ is apparent, either from theoretical considerations or from numerical experiments (done informally by Pablo Parrilo). Of course, $\mathcal{C}$ has to be a semi-algebraic set with a connected interior, which is equal to the closure of its interior.

3.2 NC Convex Sets and NC LMIs

sec:LMIrep

There seems to be a disconnect between the results of the previous section and much of the control theory LMI literature. In the literature, one often sees methods for construction which depend on Schur complement formulas and which clearly represent only very special sets with LMI's. In the previous section we saw that the relatively weak RZ condition was all that was required for there to exist an LMI representation at least in 2 dimensions. Maybe one reason is that most formulas obtained in the systems literature are noncommutative (see § 5 for why). In this section we turn to noncommutative versions of LMI representations.

Conjecture NCLMI 1 *Suppose r is a symmetric NC rational function. The positivity set*

$$\mathcal{D}_r := \{X = \{X_1, \cdots, X_g\} : \ \ X_j \ an \ operator \ on \ \mathcal{H}, \ \ r(X) \ is \ PosDef \ and \ finite\}$$

has "component" containing zero which is is a convex set iff $\mathcal{D}_r$ has a "noncommutative LMI" representation, that is there is a monic NC symmetric linear pencil L such that $\mathcal{D}_L = \mathcal{D}_r$. Recall the de£nition of a NC linear pencil L from § 2.2. Recall it has symmetric noncommuting variables x_j which are substituted by symmetric matrices in the de£nition of $L(X)$.

Conjecture NCLMI 2 *Let p be a symmetric NC polynomial. The positivity set $\mathcal{D}_p$ of p is convex only if there is a symmetric NC polynomial $\tilde{p}$ of degree 2 or less such that*

$$\mathcal{D}_p = \mathcal{D}_{\tilde{p}}.$$

This is the extension to sets of Theorem 2.1.

Example Relationship between the conjectures NCLMI1 and NCLMI2. Consider the following degree 2 polynomial $p(x)$ given by

$$p(x) := 1 + 2x_1 + 3x_2 - (3x_1 + 5x_2)(3x_1 + 5x_2). \tag{11}$$

eq:poly2

It is easy to show that the answer to Conjecture NCLMI 1 is yes for this polynomial $p(x)$. Indeed, the monic linear pencil $L(x)$ given in equation (7) and $p(x)$ satisfy $\mathcal{D}_p = \mathcal{D}_L$. To see this note that $p(x)$ is a Schur complement for the 2×2 matrix function $L(x)$ and that $p(X)$ being PSD implies $1 + 2X_1 + 3X_2$ is also PSD, which together are equivalent to $L(X)$ being PSD. Any degree 2 concave polynomial can be put in a factored form like (11). ■

McCullough, Vinnikov and Helton are making considerable progress on the conjectures, it requires building considerable operator theoretic machinery along the lines of a noncommutative real algebraic geometry. Our approach combines progress on the NC rational representation problem in §2.3 with the characterization of convex sets $\mathcal{C}$ as having de£ning function whose Hessian is positive on the tangent space to the boundary of $\mathcal{C}$ at each point. The later makes formal sense as a purely algebraic condition but an issue is proving that a NC "Zariski" tangent space is the true tangent space on a type of "NC Zariski open set". This parallels a basic piece of structure in the classical commutative case.

Question: *Also in engineering, certain simple changes of variable, typically linear fractional or bilinear, occur. Completely open is to see the realm of effectiveness of such NC transformations on NC convex sets. Helton's guess is that the NC case is better (more rigidly) behaved than the commutative case.* This seems important and work on it is actually beginning to show some promise.

4 OPTIMIZATION OVER MATRICES AND OVER SPACES OF ANALYTIC FUNCTIONS

sec:opt

4.1 Analytic Functions

The OPT problem: Given the function $\Gamma(e^{i\theta}, z) \geq 0$ of $e^{i\theta} \in \mathbf{T}$ and $z \in \mathbf{C}^N$ £nd

$$\text{(OPT)} \qquad \inf_{f \in A_N} \sup_{\theta} \Gamma(e^{i\theta}, f(e^{i\theta}))$$

where A_N is the class of smooth $\mathbf{C}^N$-valued functions on $\mathbf{T}$ which have analytic continuations on the disk. The special case $N = 1$ and $\Gamma(e^{i\theta}, z) = |g(e^{i\theta}) - z|^2$ gives the classical *Nehari problem*

(Neh) $$\inf_{f\in A} \|g-f\|_\infty$$

of £nding the distance of the smooth function g to A_1. Here $\mathbf{T}$ denotes the unit circle.

The Qualitative Theory of OPT is: Given $\Gamma(e^{i\theta}, z)$ real analytic in z, smooth in θ, with z sublevel sets which are connected, simply connected and uniformly bounded.

EXISTENCE: An optimum f^* in H^∞_N exists under a polynomial convexity assumption (which will hold for most engineering Γ) Helton- Marshall 1990, Šlodkowski.

SMOOTHNESS: Is an optimum f^* in H^∞ smooth? Yes, when $N = 1$, (Helton- Marshall) Yes, when $N > 1$ with strict convexity (Slodkowski). Otherwise OPEN when $N > 1$.

UNIQUENESS: When $N = 1$, f^* is unique (Helton-Marshall). When $N > 1$ sometimes unique, sometimes not. When $\Gamma(e^{i\theta}, z)$ is a strictly convex function in z, yes, (Helton-Howe 1986). When Γ "misses convexity by one dimension", yes, by Vityaev 1997 (my student). An elegant test for global optimality is [HWsicon].

This area is in good enough shape that Merino and I wrote a book [HMer98]. The book describes engineering control design methodology, mathematical theory, and computer algorithms.

MOPT

Given $\mathbf{\Gamma}(e^{i\theta}, z)$ a smooth, self-adjoint, $m\times m$-matrix valued function (values are in $M_{m\times m}$)

Find $f^* \in A_N$ and $\gamma^* > 0$ such that

$$\textbf{(MOPT)} \qquad \gamma^* = \sup_\theta \|\mathbf{\Gamma}(e^{i\theta}, f^*(e^{i\theta}))\|_{m\times m} = \inf_{f\in\mathcal{A}_N} \sup_\theta \|\mathbf{\Gamma}(e^{i\theta}, f(e^{i\theta}))\|_{m\times m}$$

Here $\|\cdot\|_{m\times m}$ denotes the operator norm (largest singular value.) While $\mathbf{\Gamma}$ is smooth, the matrix norm is not. And so the function $\Gamma(e^{i\theta}, z) := \|\mathbf{\Gamma}(e^{i\theta}, z)\|_{m\times m}$ is not smooth.

Example 1. The Nehari problem for matrix valued functions. is a rather special case of MOPT studied by many mathematicians including Ando, Nagy-Foias, Sarason, Adamajan-Arov-Krein, Clark and subsequently Douglas-Muhly-Pearcy, Aresene-Ceausescu-Foias, DeWilde-Dym, Glover, Foias-Frazho, Foias- Tannenbaum, Rovnayk-Rosenblum, Ball-Gohberg, Ball-Helton, and at least a thousand engineers.

Example 2 . Competing constraints such as "Multidisk" Nehari Problems.

Example 3. Solving large classes of matrix inequalities.

Example 4. Spectral Radius Nevannlina-Pick Interpolation. This was studied by Bercovici-Foias- Tannenbaum, Peller-Young, Agler-Young. This is a special case of μ-synthesis.

The problem is (in my opinion) central to a systematic theory for frequency domain design of engineering systems. Most problems with many inputs and outputs present themselves as one with a matrix valued performance measures (cf. books by Ball-Gohberg-Rodman, Doyle-Francis-Tannenbaum, Dym, Foias-Frazho, Francis, Glover-McFarland, Greene and Limebeer, Helton, Merino-Helton [HMer98]). Going to a smooth scalar valued function as in OPT is an approximation. Getting rid of this approximation besides being mathematically interesting, has the bene£t that one can go after engineering problems in a way which is exactly as they appear in engineering treatments and commercial software for doing analytic optimization (eg. μ- tools, matlab tool box).

Characterization of Optima The beginning of theory typically calls for characterization of solutions to MOPT. This is done not only in terms of the solution f^*, but also in terms of a solution Ψ to a dual problem. One gets what is called a primal dual optimality condition. For this problem it is Theorem 17.1.1 of [HM98]. The optimality conditions can be grossly abbreviated as

$$\mathcal{T}(f^*, \Psi, \gamma^*) = 0.$$

which denotes an equation on the unknown functions f^*, Ψ on $\mathbf{T}$and on the positive number γ^*.

Since $\mathcal{T}$ is smooth we can take its differential, $\mathcal{T}'$ quite explicitly. The key issue for numerical computation is the invertibility of $\mathcal{T}'$ and of variants of $\mathcal{T}'$. The motivation is that this invertibility is what determines good (second order) convergence in Newton's method and in most other methods for solving the equation $\mathcal{T}(f^*, \Psi, \gamma^*) = 0$. Different computer algorithms arise by parameterizing f^* and Ψ differently. In particular two algorithms Dym, Merino and I have recently studied are based on representing Ψ as

$$\text{Algo 1: } \Psi = G^T + G \qquad \text{or} \qquad \text{Algo 2: } \Psi = G^T G$$

where G is matrix valued and analytic on the disk. Indeed after a bit of massaging we recently found

res:toep **Result 4.1 (DHM02)** *For both algorithms the differential $\mathcal{T}'$ of $\mathcal{T}$ always has the form*

$$\mathcal{T}' = \textit{a block Toeplitz operator } + \textit{ a block Hankel operator}$$

In Algo 1 the block Toeplitz operator is selfadjoint with some useful added structure.

A consequence of the special form of $\mathcal{T}'$ in Result res:toep 4.1 is that the key step in Newton's method should be implementable with a fast linear solver (as studied by T. Kailath, A. Sayed, V. Olshevsky and others).

Result 4.2 (DH03) *For Algorithm (2.) we have a simple formula to tell if $\mathcal{T}'$ is Fredholm of index 0.*

Our computer experiments show that whether or not the Fredholm index of $\mathcal{T}'$ is 0 is an excellent predictor of the performance of Newton type algorithms. We think this actually gives the most practical approach to analyzing numerical algorithms for these problems. Computing invertibility of $\mathcal{T}'$ is likely impossible. However, index 0 computation (while a bit tricky and not that obvious) is doable on broad classes of problems and gives a theoretical tool for algorithm evaluation and development.

A corollary of these results for the multidisk problem alluded to in Example 2 is

Result 4.3 (DH03) *For a "nondegenerate" v- disk Nehari problem with $v \leq m$, the Fredholm of index of $\mathcal{T}'$ is 0 if and only if $v = m$.*

Indeed, the $v = m$ problem turns out to be well behaved for Algorithm (2.) and $v \leq m$ does not.

Currently, we are analyzing Algorthm (1.). Conspicuously open is that when $v \leq m$ and relaxed variant (an interior point method) of Newton's algorithm works well. This is a very interesting thing to analyze. Also we have in qualitative properties of multidisk problems: if all data are rational functions then for "nondegenerate" $v = m$ problems the optimum is rational. Uniqueness in the $v = m$ case is open. Also interesting is the possibility of applications of these methods inspired by primal-dual numerical approaches to other problems, like optimization for matrix inequalities, which we now describe.

sec:optmat

4.2 Optimization to Solve Matrix Inequalities

A serious effort with Camino (student) and Skelton MAE Dept. UCSD is going into numerical solution of matrix inequalities (not necessarily linear) on matrix unknowns. We preserve the matrix structure of the unknowns throughout our algorithm for as long as possible, as opposed to other semide£nite programming algorithms which at the outset express an unknown matrix, say X as $\{x_{ij}\}$, which in applications often has thousands of variables. The £rst part of the algorithm is symbolic and produces noncommutative directional £rst and second derivatives to symbolically produce a linear subproblem. Then the code feeds this to a numerical linear equation solver.

This plus our ConvexityRegion command produces an alternative strategy to LMI's which are used now throughout systems engineering. With LMI's one must actually convert a problem (if possible) to an LMI; this an art not a science. Advantages of LMI's are that their inherent convexity guarantees that solutions found are a global optima and, as mentioned before, numerical software is very highly developed. With our approach one need not know anything about LMI's. One uses NCAlgebra's ConvexityRegion command to determine a region $\mathcal{R}$ on which the problem is convex. On this region our code is reliable numerically (an experimental as well as theoretical observation) and the optimum we £nd is a global optimum. In speed comparisons our code is in its £rst incarnation a little slower than the LMI toolbox which is very highly developed.

Questions: concern comparison of this to problems tractable with LMI methods. Here it would be very interesting to know how our region of convexity $\mathcal{R}$ corresponds to the regimes described by the LMI. This is an untouched theoretical issue. Also the linear subproblem $\mathcal{L}$ our algorithm produces has at the symbolic level the special form

$$\sum_{j}^{M} a_j h b_j + b_j^T h a_j^T = q \tag{12}$$

eq:linSub

where h is the update direction which one must compute numerically at each iteration and the coef£cients a_j, b_j are NC rational functions of the basic unknowns x in the problem. One linear system of equations can have many different representations (eq:linSub 12) with many possible M. The speed of our brute force numerical linear solver decreases with M. What is the best possible M (or at least get some estimates)? Find symbolic methods for writing the linear subproblem $\mathcal{L}$ with the smallest possible M. Our current symbolic method for reducing M is sensible, often saves factor of more than 10 in time, but is crude. Finding numerical solvers which use the special structure of (eq:linSub 12) very ef£ciently would be valuable.

sec:motiv

5 LINEAR SYSTEM THEORY MOTIVATION

Matrix inequalities have come to be extremely important in linear systems engineering in the past decade. This is because many linear systems problems convert directly into matrix inequalities.

Matrix inequalities take the form of a list of requirements that polynomials or rational functions of matrices be positive semide£nite. Of course while some engineering problems present rational functions which are well behaved, many other problems present rational functions which are badly behaved. Thus taking the list of functions which a design problem presents and converting these to a nice form, or at least checking if they already have or do not have a nice form is a major enterprise. Since matrix multiplication is not commutative, one sees much effort going into calculations (by hand) on noncommutative rational functions, although engineers seldom use (don't like even) the word noncommutative. A major goal in systems engineering is to convert, if possible,"noncommutative inequalities" to equivalent Linear Noncommutative Inequalities (effectively to LMI's).

The most basic efforts, such as determining when noncommutative polynomials are positive, convex, convertible to noncommutative LMI's, transformable to convex inequalities, etc., force one to the rich area of undeveloped operator and matrix theory described in this proposal.

Many different types of matrix inequalities have come up in the mathematics of the previous century, but the ones which predominate in engineering systems usually take the form of a polynomial or rational function of matrices being positive semide£nite. An extremely simple example is the Riccati inequality

$$AX + XA^T - XBB^TX + C^TC \;\; is \; PosSemiDef. \tag{13}$$

ineq:ric

Also there is the LMI

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \;\; is \; PosSemiDef. \tag{14}$$

ineq:lmi

The inequalities (13) and (14) are equivalent in that given matrices A, B, C they have the same set of solutions X. Note (14) is linear in the unknown X; thus is an LMI. It is algebraic formulas like these (though typically more complicated) that are programmed into the main computer packages in engineering.

A user of one of these packages when doing a design puts in the math model for his system, that is, he gives speci£c matrices A, B, C. Numerical software in the package then solves for X.

Thus to produce design software there are two main issues.
(1) Algebraic: complicated inequalities involving polynomials and rational functions occur, convert them to nice ones or prove this impossible.
(2) Numerical: Find numerical methods for solving nice ones.
Most of the research proposed here is on such algebraic (rather than numerical) issues.

5.1 To Commute or Not Commute: "Dimensionless" Formulas

This section discusses two different ways of writing matrix inequalities. As an example, we could consider either the Riccati inequality (13) or the equivalent LMI in (14). Let us focus on this LMI, and discuss the various ways one could write this linear matrix inequality.

The LMI in (14) has the same form regardless of the dimension of the system and its de£ning matrices A, B, C. In other words, if we take the matrices A, B, C and X to have compatible dimension, (regardless of what those dimensions are), then the inequality (14) is meaningful and substantive and its form does not change.

When the dimensions of the matrices A, B, C and X are speci£ed it is common to write (14) as a linear combination of known matrices $L_0, L_1, \ldots, L_g$ of dimension $d \times d$ in unknown real numbers $s_1, \ldots, s_g$:

$$L_0 + \sum_{j=1}^{g} L_j s_j \; is \; PosSemiDef \tag{15}$$

ineq:Ugh

For example, in the inequality (15) if $A \in \mathbf{R}^{2\times 2}, B \in \mathbf{R}^{2\times 1}, C \in \mathbf{R}^{1\times 2}$, then $X^T = X \in \mathbf{R}^{2\times 2}$ and we would take $m = 3$ and the numbers s_i in $X = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$ as unknowns in the inequality (15). The unpleasant part is that the L_i are $L_0 := \begin{pmatrix} C^TC & 0 \\ 0 & I \end{pmatrix}$ $L_1 := \begin{pmatrix} 2a_{11} & a_{21} & b_{11} & b_{12} \\ a_{21} & 0 & 0 & 0 \\ b_{11} & 0 & 0 & 0 \\ b_{12} & 0 & 0 & 0 \end{pmatrix}$

$$L_2 := \begin{pmatrix} 2a_{12} & a_{11}+a_{22} & b_{21} & b_{22} \\ a_{22}+a_{11} & 2a_{21} & b_{11} & b_{12} \\ b_{21} & b_{11} & 0 & 0 \\ b_{22} & b_{12} & 0 & 0 \end{pmatrix} \quad L_3 := \begin{pmatrix} 0 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & a_{22} \\ 0 & 0 & 0 & b_{21} \\ a_{12} & a_{22} & b_{21} & 2b_{22} \end{pmatrix}.$$

Now consider $A \in \mathbf{R}^{3\times 3}, B \in \mathbf{R}^{3\times 2}, C \in \mathbf{R}^{2\times 3}, X \in \mathbf{R}^{3\times 3}$. This gives a messier formula. The point is that the formula (15), with commutative unknowns, does not scale simply with dimension of the matrices or of the system producing them, while formula (14) does, but (14) contains noncommutative unknowns.

Helton sees problems as splitting into two natural types **dimensionless**, the dimension of the system does not directly enter the statement of the problem, and **dimension dependent**. Most classical systems problems are dimensionless, eg. the H^2 control problem, H^∞ control problem, state estimation problems, etc.. It is an empirical observation that dimensionless problems convert to matrix inequalities in noncommutative variables, while those which are dimension dependent lose this structure and have commutative variables. For example, the H^2 control problem converts to solving one Riccati inequality, while the H^∞ control problem converts to solving two Riccati's and a coupling inequality; all of these are inequalities on polynomials in noncommutative variables.

6 NONCOMMUTATIVE COMPUTER ALGEBRA

sec:ncalg

Operator theory and linear systems engineering are rife with calculations in a noncommuting algebra. Indeed it is likely that someday there will be a branch of these subjects devoted to their symbolic computational aspects. While commutative computer algebra has seen heavy development and use, since the MACSYMA project in the 1960's, general noncommutative computer algebra has only recently come to the beginning stages of experimentation; still the £eld is uncharted, certainly with respect to engineering and operator theory problems. For perspective, 10 years ago there was little noncommutative algebra software publically available. Unfortunately, to bring noncommutative computer algebra to nearly its potential requires a creation of a small world of algorithms and software. Also the £eld, to get started, faces the task of doing many experiments to £nd a collection of successful applications and of developing more algorithms to £ll major application gaps.

Our software. Helton-Stankus plus many students are the main providers to Mathematica of platform (called NCAlgebra) for performing general noncommuting calculations in Mathematica. It has over a thousand downloads. Our effort, which includes many features, now consists of 1.8 Megabytes of Mathematica code with 2.14 Megabytes of C++ linked. See the NCAlgebra homepage, *www.math.ucsd/ ~ ncalg.*

The pure Mma part of NCAlgebra can be used as a very powerful 'yellow pad', and runs on most platforms. In the last year we have implemented algorithms involving noncommutative inequalities, for example the convexity checker mentioned in §2.4 These include a noncommutative LDU decomposition.

The noncommutative Gröbner basis, NCGB part of NCAlgebra, implements algorithms due to Mora in C++ and links to Mathematica. We have algorithms for sorting and "shrinking" the output in various ways (this is crucial for noncommutative situations). It supports Solaris; Windows; and Linux.

6.1 Experiments on Inequalities

sec:expInq

Progress has been fast in the areas proposed in §1, 2, 3 due partly to Mathematica experiments. For example, these helped vet the Sum of Squares results, and our NC Algebra command ConvexityRegion and related software suggested the extremely rigid structure of convex polynomials and convex rational functions. Also they vetted the proof outlines we are now following on noncommutative rational functions and NC LMI representations. In fact most of our experiments in the last year have been directed at these questions.

Recently graduate students Slinglend and Shopple came up with a new algorithm and NC Algebra implementation of it for producing representations for rational functions as in (8). This will allow much more penetrating experiments and even the possibility of producing an LMI representation algorithmically.

6.2 Noncommutative Gröbner Basis

Noncommutative Gröbner Bases are experimental, but we £nd them a valuable tool. They can be used to eliminate variables from systems of polynomial equations. NCAlgebra is linked with NCGB, an implementation of Mora's algorithm, developed by Mark Stankus my former postdoc. We have done many experiments and continue to. NCGB was quite useful in work with M. Oliveria described below. We found that NCGB plus our "throwing out redundant relations" methods, when tried on singular perturbation problems as one sees in linear control, works well, see [HKMS00]. We are planning to try NCGB in addressing the problem of making N as small as possible in the linear subproblem (12). Understanding the behavior of noncommutative GB for pseudoinverses and sums of several pseudoinveses would have serious implications if they behave simply. We have done numerous experiments and tried the results with mixed results on the LMI producing method of R. E. Skelton, T. Iwasaki, K. Grigonidas [SIG97]

Stankus has recently made a major overhaul of his NCGB code, producing big speed improvements, and we plan over the next few years to link this new version to NCAlgebra. Of other noncommuters, Ed Green at VPI has been helpful to us, as was his former postdoc, Ben Keller, who wrote the excellent noncommutative GB code OPAL. The only other noncommutative GB codes we know of, *http://www.singular.uni-kl.de/DEMOS/PLURAL/overview.html* and *http://www.win.tue.nl/ ~ amc/pub/grobner/doc.html,* are from Europe and we are not sure of their scope (or the extent to which they will be supported).

sec:compMisc

6.3 Miscellaneous

Computer Algebra for Standard LMI Methods One venture with Mauricio Oliveria, a postdoc in MAE, concerns the unifying 1997 book [SIG97] on control vs. linear matrix inequalities. We, in [OH03], have developed algorithms (mostly factorization) which allow implementing the method in this book. There is another well known method "the change of variables approach" by C. W. Scherer, P. Gahinet and M. Chilali in 1997. Here much can be implemented and we are doing experiments on the tricky change of variables involved, [OH03cdc]. They seem to £t the mold of a rather modest change of variables procedure proposed in [HS99]. Methodology so far is implemented in NCAlgebra, although not distributed yet. Parts of all of this involving automatic production of pseudoinverses provide interesting challenges.

Mathematica's Control Toolbox `Mathematica` has a control toolbox which operates at the commuting level. Thus it will not deal with block systems. We have and plan to maintain a small package under `NCAlgebra` which links it to `Mathematica`'s package and gives it a reasonable amount of noncommuting capability. This requires no change to `Mathematica`'s package just load our package in. Then one can manipulate the A, B, C, D of systems theory much as a human would do. See Synergistic Activities in this proposal.

sec:nl

7 NONLINEAR OPERATOR THEORY AND CONTROL

The last main topic in the proposal is a major interest but will receive only brief treatment, since we are out of space.

In the linear case there are many equivalent formulations of the problems and results described in the proposal. They can be presented in several very different terminologies such as modern operator theory and classical complex variables, control engineering, classical circuit theory. It is an overwhelming job to give them all in this proposal, so we typically present only one formulation and urge the reader to use some imagination in converting the result to the language most familiar to him. Outside the linear realm what result is equivalent to what is an interesting research topic, but still the correspondences are strong.

The nonlinear area is open ended and results with James are already extensive enough that we are forced to write a monograph [HJ99] in order to have a complete account of our results. The most dramatic results and ideas of proofs were announced in Conf. on Decision and Control proceedings.

It would be appropriate to start gently with something fun like the non-linear generalization of the analytic function maximum principle. But in a 15 page sprint I must go directly to heavier things.

The problem After manipulation, the H^∞ control problem reduces to the problem involving the following £gure:

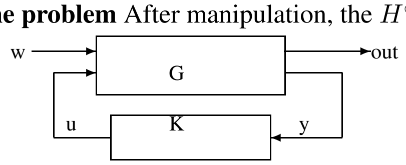

Figure CTRL.

The given system G, described by state space equations

$$dx/dt = \mathcal{A}(x) + B_1(x)w + B_2(x)u, \quad out = C_1(x) + D_{12}(x)u, \quad y = C_2(x) + D_{21}(x)w,$$

is nonlinear but time-invariant. We always assume homogeneity, i.e. $\mathcal{A}(0) = 0,\ C_1(0) = 0,\ C_2(0) = 0$. One wishes to £nd a nonlinear time-invariant feedback system K of the form

$$dz/dt = a(z, y), \quad u = c(z)$$

which achieves given performance γ. The standard problem of H^∞ control in the nonlinear setting is

(CTRL) *Find a feedback law (a, c) so that the resulting closed loop system when initialized at 0 satis£es the dissipation inequality*

$$\int_{t_0}^{t_1} \|out\|^2\, dt \le \gamma^2 \int_{t_0}^{t_1} \|w\|^2\, dt$$

for a preassigned tolerance level γ and which makes $x(t) \to 0$ regardless of initial state.

In the case of linear systems

$$\mathcal{A}(x) = \mathcal{A}x, \;\; B_j(x) = B_j, \;\; C_j(x) = C_jx \;\text{ and }\; D_{ij}(x) = D_{ij}$$

where $\mathcal{A}, B_j, C_j, D_{ij}$ are matrices and the feedback system is linear as well. The frequency response functions of the systems (which describe the input output behavior of the systems) are rational matrix valued functions. The question (CTRL) and related questions can be recast in terms of frequency response functions (FRF) and become complex variable problems of a type studied by operator model theorists. The special case of (CTRL) where the given system is stable and the $u \to y$ FRF is 0 is called the model matching problem in engineering, or the (matrix valued) Nevanlinna-Pick (NevP) problem (with a £nite number of interpolating conditions) in mathematics.

CTRL also is the problem of making a circuit dissipative using feedback. For linear systems key cases were solved in 1965 (SISO) by Youla and Saito and (MIMO) in 1976 by Helton. In the early 80's Zames and Francis formulated H^∞ control and solved the math problem by drawing on the earlier solutions to this circuits problem. In the beginning the subject of H^∞ control evolved quickly in signi£cant part because key math problems were already reasonably understood by operator theorists.

Parameterization of all linear H^∞ solutions to (CTRL) can be done once one expresses the given system in Fig CTRL in certain coordinates and writes it as the composition of two systems with special properties; this amounts to a " J-inner/outer" factorization of a given operator on vector valued $L^2[0,\infty]$. Mathematically this is a vast extension of the Beurling Theoerm which says that functions analytic on the right half plane (RHP) have an inner outer factorization and this is the general area which the Nagy-Foias commutant-lifting theorem addresses (following a basic insight of Sarason). The £rst solution to linear (CTRL) was done not in statespace form as presented here but in terms of FRF's, using commutant lifting solutions to the matrix Nevanlinna-Pick problem by Zames and Francis. It followed my suggestion to Zames. Then K. Glover gave a solution in statespace terms. Ordinary inner/outer factorization is classical Beurling -Lax theory. It was Ball - Helton who showed that " J-inner/outer" factorization had the powerful properties indicated above.

Older nonlinear work The formulation of, and £rst work on nonlinear (CTRL) was by Ball-Helton-Foias-Tannenbaum in 1987; and actually focused on the factoring problem. This was done in terms of Taylor series expansions of the system's input output operators and it did not use statespace. Foias-Tannenbaum with Gu ultimately developed this into a rich and complete theory.

The control and factorization problem for large classes of stable nonlinear systems was reduced to a Hamiltonian-Jacobi-Bellman-Isaacs (HJBI) inequality by Ball and Helton in the late 1980's. This gave a nonlinear theory which was a strict generalization of the statespace linear theory. Work of Byrnes and Isidori around 1992, making a system stable with feedback, was at the same level of generality. Ball and van der Schaft gave a beautiful theory of nonlinear Wiener-Hopf factorization. This gives a way of producing J-inner/outer factors in the nonlinear case. Unfortunately, all of these constructions and methods work only in the stable case. In the mid 1990's the general (unstable) theory has made a big advance due to work of James-Baras- Helton, Didinsky-Basar, and also van der Schaft. The problem of extending H^∞ control to a nonlinear system has now been converted to solving two particular types of PDEs. One PDE, **the HJBI**, is the key to understanding the situation where A is stable. It is

$$\nabla_x V(x) \circ [A(x) - B(x)D(x)^{-1}C(x)] + \frac{1}{2}\nabla_x V(x) B J_U B^T \nabla_x V^T(x) = 0 \tag{16}$$ eq:HJBI

and V is called stabilizing provided the differential equation $\dot{x} = [A + BJ_U B^T \nabla_x V^T](x)$ is asymptotically stable. The other equation is new and it is scalled **the information state PDE, ISPDE**:

$$\dot{p}_t = f(p_t, s(t)), \tag{17}$$ is-dyn

where $f(p,s)$ is the differential operator

$$f(p,s) := -\nabla_x p \cdot (A + Bs) \; + \; \tfrac{1}{2}\, |C + Ds|^2_{J_Q} \tag{18}$$ f-def

Under many assumptions the control problem *(CTRL) having a solution is close to the function*

$$p_t(x) + V(x), \tag{19}$$

de£ned on $x \in \mathbf{R}^n$ the space of states, being well behaved and having a unique maximizer. Here V is the stablizing solution to the HJBI. This and improvements is the subject of the research monograph [HJ99]. The papers [HJMprept] give formulas which replace the ISPDE with a related one on a lower dimensional space, thereby saving computational effort.

At this time conspicuously missing is how the matrix inequalities which revolutionized linear systems extend to the nonlinear case. For the HJBI equation such extension is standard, but not for the IS PDE.

More speci£cally, the main question I am working on (with James and his postdoc Huang) is the "low order controller problem". This is (CTRL) but with the dimension of the state space of the controller prespeci£ed, say d. In the linear case there is a elegant solution, see Ch 8 R. E. Skelton, T. Iwasaki, K. Grigonidas [SIG97]. There are two decoupled Riccati inequalities which must have solutions X Y both PosSemiDef respectively. These must satisfy the *coupling condition* $Y^{-1} - X$ is NegSemiDef and it has rank $\leq d$. This problem will require us to understand the HJBI inequality, the information state inequality together with how the coupling condition behaves. I am optimistic about our approach, it is to the place where we are doing numerical tests of our conjectured formulas.

For perspective, a classical analytic function problem this generalizes is: Given a rational function f of one variable analytic in the closed RHP. Find a rational function g analytic in the closed RHP with d or fewer poles whose supremum norm distance to f is less than or equal to γ. This is a highly nonconvex problem. The matrix inequality solution mentioned above (handles this classical problem) and the solution converts to LMIs except for the rank condition. A signi£cant engineering effort is devoted to research on compromise solutions in the presence of such rank constraints.

8 OTHER

sec:other

To my delight UCSD just built a strong group in control. Bob Skelton was the £rst of the new wave and we already are working on several projects. One is developing computer algebra for facilitating manipulation of matrix inequalities was described in §sec:optmat 4.2 and §sec:convcheck 2.4.

Another, involves tensegrity structures. These make heavy use of cables, since pound for pound cables are stronger than steel beams. This seems like a rich area for theory and has not been extensively developed. Mathematically, these produce elegant generalizations of static structure problems where one has ”signed objects” rather than positive ones. Our £ndings so far are in [SHAPC01] and various conference proceedings.

Ed Bender, Bruce Richmond and I solved a combinatorics question about the frequencies of sequences of ”ups and downs” in strings of integers. This uses the Krien-Rutman Theorem plus some explicit calculations with integral operators [BHRprept]. Here a nice open question is what happens for a certain more general class of sequences, see Conjecture 1 [BHRprept]. It suggests a novel extension of the classical Krein Rutman Theorem.

Local Navy researchers and I discuss uses of the techniques in §sec:opt 4; this led to US. patent [SHA03] on an algorithm.

My former Phd student Mike Walker is the main controls person for fusion research here in San Diego (biggest tokamak in the U.S.). we discuss plans for future control systems.

Also Ford Motor Co. has given several gifts to support my research.

9 PERSPECTIVE

Motivation §sec:motiv **5** Matrix inequalities have recently come to dominate linear systems theory. These lead to two natural types of algebraic problems: those with variables which are noncommutative and those with commutative variables. These correspond physically to whether or not the problem is or is not ”dimensionless”. Convexity is a major issue because ultimately numerical methods are optimization based. LMI’s play a dominant role now in systems algorithms and software; at least a thousand papers concern them. The state of the engineering art is: there are clever tricks for producing LMI, but little that is systematic and in many problems MIs but no LMIs emerge. Consequently, the algebra underlying both noncommutative and commutative inequalities needs a thorough investigation. Though noncommutative ”inequalities” are more common in classical control or systems books, a theory is just beginning. The situation at the moment is

Noncommutative Semialgebraic Geometry, §sec:linOp **1** This operator theoretic version of the classical branch of real algebraic geometry called semialgebraic geometry has emerged in the last 3 years. Its status is

1. SoS and Theorem thm:sphere 1.4 behave better in the NC than in the commutative case.

2. Our NC strict Positivstellensatz behaves comparably to the Putinar-Vasilescu 1999 version of the classical commutative Positivstellensatz (which goes back to the 1970’s).

3. Characterization of the radical ideal behaves badly in the noncommutative case (as opposed to the commutative case). Various nullstellensatzs are possible. There are many open questions in non-strict cases.

4. A noncommutative version of the Tarski’s elimination of quanti£ers principle (which predates the Positivstellensatz) has yet to be explored.

NC Convexity §sec:lin.1.2 **2** When the variables are noncommuting the structure is extremely rigid. Nothing like this holds for commuting variables.

Convexity vs LMIs §sec:LMIrepCom **3**

1. For commutative variables convex sets are not all representable; for dimension $g = 2$ precise conditions are known, while for $g > 2$ they are conjectured. Possibly ”lifts” of convex sets have LMI representations.

2. We conjecture that ”noncommutative convex sets” all have noncommutative LMI representations. Indeed for noncommutative situations we believe that LMIs and convexity are intimately linked. There is considerable progress toward proving it. Changes of variable to achieve NC convexity is an open frontier, but may well be better behaved than what one sees classically in several complex variables.

NC Computation §sec:ncalg **6** Noncommutative algebraic algorithms can be readily implemented using NCAlgebra which runs under Mathematica. We do research on algorithms, implementation, experiments and we support NCAlgebra.

For noncommutative problems, numerical algorithms which completely bypass LMIs are in being tested, see §sec:optmat 4.2 Our design methodology potentially has the advantages of LMIs (including guarantees of global optimality because of our noncommutative convexity checker) without the headaches and uncertainty of trying to convert ones problem to LMIs.

Nonlinear operators and systems §7 sec:nl An achievement of the last ten years has been to elegantly extend substantial pieces of linear theory to nonlinear operators and systems. Now a frontier is extending the linear inequality theory of systems to nonlinear situations. Much remains here.

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Many people and their work were referenced in the research plan, so no formal bibliograghy is attached. Helton's papers produced in the previous £ve years are listed below.

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E. BIOGRAPHICAL SKETCH

Vita

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Positions Held: Associate UC San Diego 1974 - 1979, Full Professor UC San Diego 1979 - Present, Visiting Assoc. Professor UCLA 1974, Assistant and Associate Professor SUNY-Stony Brook 1968 - 1973

Five Papers Closely Related to the Proposal

[H02],[CHSY03], [HMtrans], [HVprept] and [HJ99]

Five Signi£cant Publications

I gather we are supposed to list £ve in¤uential papers here. To give a ¤avor of my work before doing mathematics connected with control (heavily discussed earlier in the proposal) I decided to pick 5 earlier papers written before 1980.

[H72] J. W. Helton: "In£nite dimensional Jordan operators and Sturm-Liouville conjugate point theory," Trans. Amer. Math. Soc., 170 (1972), 305 331.

This gives an unexpected lifting theory for a class of seemingly abstract operators de£ned algebraically. It turns out to generalize classical O .D. E. theorems. This was the forerunner of Agler's very powerful hereditary operator theory.

[H74] J. W. Helton, Discrete time systems, operator models and scattering theory, J. Functional Analysis, **16** (1974), 15–38.

This was the discovery that operator models and engineering systems theory are essentially equivalent. Independently P. Fuhrmann and P. DeWilde observed something similar. It lead to a branch of operator theory which interacts with engineering.

[HH76] J. W. Helton and R. Howe, Traces of commutators of integral operators, Acta Math., **136** (1976), 272–305.

This was the £rst paper which revealed surprisingly concrete and elegant structure in this area. The paper in¤uenced Alain Connes in his invention of noncommutative differential geometry. (Howe and I did the highest cohomology class in Connes cyclic cohomology.)

[H77] J. W. Helton, An operator algebra approach to partial differential equations; propagation of singularities and spectral theory, Indiana J. of Math. (1977), 997–1017.

Gives a basic result on the spectrum of the Laplacian on any manifold whose geodesics have uniformly bounded length.

[H78] J. W. Helton, Orbit structure of the Möbius transformation semigroup action on H^∞ (broadband matching), Advances in Math. Suppl. Studies, **3**, Academic Press, new York (1978), 129–197.

This found the Poincare distance of a function in BL^∞ to BH^∞. It also solved the long standing circuits problem of multiport broadband gain equalization. This work introduced commutant lifting and AAK techniques to engineering. H^∞ control is where these found their biggest use. H^∞ type packages now dominate linear control systems software.

Synergistic Activities

1. I devote much effort to promoting interactions between operator theorists and engineers. One is by doing research at the boundary of the subjects. Another is through conferences. I think that ' 'pure" mathematicians, applied mathematicians, and engineers, should meet from time to time, at the minimum for educational purposes and at the maximum for serious collaboration. Thus I think having some broad ranging conferences is valuable. With Gohberg, I founded the conference International Workshop on Operator Theory and Applications (IWOTA) in 1981. Gohberg, Kaashoek and I constitute the central committee and have managed to sustain it. Also I am an originator and (a steering committee) mainstay of the Mathematical Theory of Networks and Systems. Both meet every 2 years - IWOTA year 2000 meeting had 170 participants- MTNS had 950. IWOTA year 2002 meeting had 100 participants- MTNS had 400.

2. Mark Stankus and I together with students provide (for free) the noncommutative computer algebra package, NCAlgebra, which gives Mathematica general noncommutative capability. It has had over a thousand downloads. See §sec:ncalg 6 for descriptions of related research. We have good ties with Wolfram Research Inc., for example, Mathematica has a control toolbox (by I. Bakshee) which of course operates at the commuting level. Thus it (or any other package) will not deal with systems in a way similar to the way a human deals with systems. We wrote a small package which links it to NCAlgebra with some advice from I. Bakshee which gives Mathematica's *Control System Professional* a reasonable amount of non-commutative capability. I think this is a £rst for systems packages. Our enhancement is distributed free with NCAlgebra. See the NCAlgebra home page, www.math.ucsd/ ∼ ncalg.

3. Teaching- Summer graduate student computer lab (see Human Resources) Students learn more math, gain computing skills and get a taste of engineering. I am developing an Undergrad "taste of engineering" math course. The prerequisite is linear algebra and complex analysis. The material is not close to any course I have seen.

4. Book with Merino [HMer98] gives a uni£ed engineering control design methodology, the corresponding mathematical theory developed over the last 10 years, computer algorithms, and some software implementing our algrithms. About 60 % of the book is very expository and attempts to achieve a balance between mathematics and engineering which can serve as an introduction to both audiences.

5. Associate Editor: Journal of Operator Theory, Journal of Operator Theory and Integral Equations, Nonlinear and Robust Control, CRC book series, Journal of Fourier Analysis and its Applications, Birkhauser book series in Control, SIAM book series: Advances in Design and Control.

Collaborators over the last 48 months

J. Agler, Math UCSD; Adhikari, R. E. unknown; E. Bender Math UCSD; J. F Camino, Wailung Chan, Gr Stu Eng UCSD; P. Dower, Eng Melborne U; H. Dym, Math Weitzman Inst.; M. Hardt, Math Tech U Darmstadt; M. R. James, Eng ANU; K. Kreutz-Delgado, Eng UCSD; F. D. Kronewitter, Titan Inc.; W. M. McEneaney, Math & MAE UCSD ; O. Merino, Math URI; M. Oliveria, postdoc MAE UCSD; M. Putinar, Math UCSB; B. Richmond, math U Waterloo; Lev Sakhnovic Courant Inst Vis Scholar; R. E. Skelton, Eng. UCSD; M. Stankus, Math Cal Poly San Luis Obisbo; O van Stryk, Math T U Darmstadt; V. Vinnikov, Math BG U of the Negev; T. Walker, Software consulting; J. Wavrik, Math UCSD; M. A. Whittlesey, Math Cal State San Marcos; J. Ye, in Singapore; S. Yuliar, A Univ. in Indonesia;

My Graduate Adviser Mike Crandall 1968

Graduate and Postgraduate advisees in last 5 years

Postdoctorals— P. Dower- Eng. Melborne U. , Matthew Kennel, Mikhail Sushchik- Inst Nonlin Studies UCSD, Andres Balogh - Math Dept. UTexas Pan Am

Graduate students — who were Phd advisees, or substantial coauthors with me and who recieved some funding are: D. Kronewitter - Titan Inc. M. Hardt - TU-Darmstadt (Postdoc Applied Math).

Additional Grad Students funded from various Helton grants (most worked on various computational and engineering related projects.)– In math: Brett Kotschwar, Poon Chuan Adrian Lim, Karl Hakan John Shopple, Nicholas Slinglend, Anthony Mendes, Jeffrey Scott Ovall, Anthony Shaheen, Jason Bell, Josh Grif£n, Daniel Curtis, Jieping Ye, Dave Glickenstien, E. Rowell, In engineering: Juan Camino, Michael Hardt.

Undergrads Mike Torre, Maria Campbell

Often these math grad students get Phd's in pure math so, the lab experience and association with the engineering students broadens them considerably.

BUDGET JUSTIFICATION

Salaries.

The request is for one month of summer salary for the PI, for some postdoc, graduate student and undergraduate support. Besides graduate students who work directly with me, many do computer and related projects. Some of this is described in §6 on NCAlgebra. Also computer experiments with graduate students played a critical role in finding Theorems 0.1, 0.2, 1.1 as well as the Conjectures in Sections 2.3, 3.2 and research in Sections 4 8.

Foreign travel. I usually go to between 2 to 4 conferences per year. Often one is international the others are domestic. I usually go to an IEEE conference , MTNS, IWOTA, and attend individual operator theory conferences or other conferences as they come up.

Equipment. I could sure use a faster computer with fast access memory for our computer algebra calculations. Cost is $5000 in year 1 only.

G. CURRENT AND PENDING SUPPORT

NATALIE WILL PROVIDE

H. FACILITIES, EQUIPMENT AND OTHER RESOURCES??

I. SPECIAL INFORMATION AND SUPPLEMENTARY DOCUMENTATION??