November 25, 1995

Dissipative Control Systems Synthesis with Full State Feedback

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November 25, 1995

Abstract

This paper concerns a general dissipative control problem for nonlinear systems with full state feedback. This problem includes positive real control and H_∞ control problems as special cases. The paper shows that the solution to the control problem is equivalent to the existence of a type of solution to a partial differential inequality (nonlinear systems case) or an algebraic Riccati inequality (linear systems case) and stability results are obtained under appropriate detectability assumptions. That is the main pursuit of the paper. In addition, we obtain a local smooth solution to the PDI based on the solution for the linearized systems. Finally, we consider an application of the results to a robust stabilization problem.

Key words: Dissipative control, nonlinear and linear systems, dynamic programming equation, viscosity solutions, nonlinear robust control.

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1 Introduction

In recent years there has been a renewed interest in the study of finite gain and positive real systems and their synthesis (see for example [29], [7], [6], [20], [22], [8], [10] in the nonlinear case, and [23], [13], [3], [15], and [26] in the linear case). For linear systems H_∞ is an effective attempt at codifying the area of classical control, which while dominating industrial practice, had always been done purely by trial and error. Related to this has been the need to obtain a control system which is robust with respect to uncertainties. In the robust control problem, one deals with a system interconnection which consists of a nominal system and a description of uncertainties. By utilizing Nyquist's stability criterion one can establish the stability of interconnection of either finite (small) gain systems or positive real systems. Thus, by modeling uncertainties as finite gain or positi! ve real systems, one may design a finite gain or positive real nominal system that is robustly stable with respect to the uncertainties. In a recent paper [14], the sector boundedness concept has also been proposed to achieve less conservative robust control systems.

Finite gain, positive real, and sector bounded concepts are a few special cases of the more general concept of dissipativity which is important in the stability analysis of dynamical systems. Central to the concept of dissipative system is the supply rate function $r(z, w)$, where z and w are the system output and input respectively ([30], [31], [16], [17]). A system is called dissipative if

$$\begin{cases} -\int_0^T r(z(t), w(t))dt \ \leq \ \beta(x_0), \\ \forall w, \forall T \geq 0, \end{cases}$$

for some finite function β. In [30], [16], [17], it is shown that dissipative systems posses a set of energy-like functions (called storage functions) which, under detectability assumptions, serve as Lyapunov functions and, therefore, ensure the stability of the systems.

The synthesis problem for finite gain (or H_∞), or positive real control sytems have been addressed independently, either in the state feedback or output feedback case. Finite gain control synthesis problem is addressed using various techniques in [29], [7], [6], [20], [22], [23], and [13], to name but a few, whereas the positive real control synthesis is presented in [8], [10], [15], [26]. In a recent paper [32], a method is proposed for the synthesis of general dissipative control systems using output feedback. The dissipativity includes finite gain, positive real, and a mixture between finite gain and positive real performances as special cases.

The purpose of this paper is two fold. It presents results analogous to those in [32] for nonlinear and linear systems with full state feedback. Secondly, the HJBI inequalities crucial to nonlinear state feedback control are typically treated locally or under the assumption that solutions are smooth. Papers [5], [21], [11] and [25] deal with a type of weak solution called *viscosity solution*. In this paper we give global theorems extending this literature.

We describe the problem formulation in §2. In §3 we present some basic results concerning (strictly) dissipative systems. In §4 we express the solution to the problem in terms of a partial differential inequality (PDI), and deduce stabilty under appropriate closed loop detectability assumptions. In the linear case, this PDI reduces to an algebraic Riccati inequality (ARI). In §5 we obtain local smooth solution to the PDI based on the solution for the linearized systems. This extends the known results in the H_∞ control case in [27], [28], [29]. We apply the synthesis results to a general robust stabilization problem in §6. Results related to viscosity solutions are presented in the Appendix.

2 Problem Formulation

We consider a general class of nonlinear systems described by

$$\Sigma : \begin{cases} \dot{x}(t) &= A(x(t)) + B_1(x(t))w(t) + B_2(x(t))u(t), \ x(0) = x_0, \\ \\ z(t) &= C_1(x(t)) + D_{11}(x(t))w(t) + D_{12}(x(t))u(t). \end{cases} \tag{2.1}$$

In this description, $x \in \mathbf{R}^n$ denotes the state vector, which is available for control. The vector $z \in \mathbf{R}^q$ represents the quantity to be controlled. The disturbance $w \in \mathbf{R}^d$ corrupts the state and the vector $u \in \mathbf{R}^m$ denotes the control. We assume that $A(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$, $C_1(\cdot)$, $D_{11}(\cdot)$, $D_{12}(x)$ are smooth functions, $A(\cdot)$ and $C_1(\cdot)$ are globally Lipschitz continuous, $B_1(\cdot)$, $B_2(\cdot)$, $D_{11}(\cdot)$ and $D_{12}(\cdot)$ are bounded, and that $A(0) = 0$, $C_1(0) = 0$. We further assume $D_{12}(x)'D_{12}(x) = E_2(x) > 0$, for all x.

The admissible class of control is the set of static functions of the state $\mathbf{u} = K(x)$ with

$$K : \mathbf{R}^n \to \mathbf{R}^m.$$

We denote the class by $\mathcal{S}$. Admissible disturbances are all signals $w \in L_2([0,\infty), \mathbf{R}^d)$.

We consider the general supply rate $r(z, w)$

$$r : \mathbf{R}^q \times \mathbf{R}^d \to \mathbf{R}, \tag{2.2}$$

which is assumed to be $C^1(\mathbf{R}^q, \mathbf{R}^d)$, with at most quadratic growth, i.e. $|r(z, w)| \leq \alpha(1 + |z|^2 + |w|^2)$ for some $\alpha > 0$, for all $z \in \mathbf{R}^q$, $w \in \mathbf{R}^d$. Throughout the paper, we consider supply rate functions that satisfy

$$r(z, 0) \leq 0, \tag{2.3}$$

for all $z \in \mathbf{R}^q$, with $r(0, 0) = 0$. Let $\Sigma_{x_0}^{\mathbf{u}}$ denote the map from w to z, under control strategy $\mathbf{u}$, with initial condition x_0.

Definition 2.1 *We say that the system $\Sigma^{\mathbf{u}}$ is* dissipative *with respect to the supply rate $r(z,w)$ if for each initial condition $x_0 \in \mathbf{R}^n$ the map $\Sigma^{\mathbf{u}}_{x_0}$ is* dissipative *with respect to the supply rate r, which means*

$$-\int_0^T r(z(t),w(t))dt \;\leq\; \beta^{\mathbf{u}}(x_0), \tag{2.4}$$

for all $w \in L_2([0,T],\mathbf{R}^d)$, for all $T \geq 0$, for some $\beta^{\mathbf{u}} \geq 0$, with $\beta^{\mathbf{u}}(0) = 0$. We say that $\Sigma^{\mathbf{u}}$ is strictly dissipative *with respect to the supply rate $r(z,w)$ if the inequality in (2.4) is replaced by*

$$-\int_0^T r(z(t),w(t))dt \;\leq\; -\frac{\epsilon}{2}\int_0^T w'(t)w(t)dt + \beta^{\mathbf{u}}(x_0), \tag{2.5}$$

for some $\epsilon > 0$.

The *dissipative control problem* is to find $\mathbf{u} = K(x) \in \mathcal{S}$ such that :

(i) the closed loop system $\Sigma^{\mathbf{u}}$ is asymptotically stable, and

(ii) $\Sigma^{\mathbf{u}}$ is dissipative with respect to the supply rate $r(z,w)$ in (2.2).

3 Dissipation PDI and ARI

The purpose of this section is to review and extend some basic results concerning dissipative systems. We consider the class of nonlinear systems described in (2.1) with $u = 0$, i.e.

$$\Sigma^0 : \begin{cases} \dot{x}(t) &=& A(x(t)) + B_1(x(t))w(t), \;\; x(0) = x_0, \\ \\ z(t) &=& C_1(x(t)) + D_{11}(x(t))w(t). \end{cases} \tag{3.1}$$

3.1 General Dissipative Systems

It is well known that dissipative systems posses a set of energy-like functions called storage functions, which satisfy dissipation type inequalities. For nonlinear systems with general (non quadratic) supply rate these inequalities take the form of the following partial differential inequality (PDI) (see [30], [16], [28], [21])

$$\sup_{w\in\mathbf{R}^d} \{\nabla_x V(x)\cdot(A(x) + B_1(x)w) - r(h(x,w),w)\} \leq 0, \tag{3.2}$$

where $h(x,w) = C_1(x) + D_{11}(x)w$, and for linear systems with quadratic supply rate, they reduce to algebraic Riccati inequalities (ARIs) (see [31] and (3.4) below). Furthermore, under appropriate detectability assumptions, dissipative systems are asymptotically stable.

Definition 3.1 *We say that* Σ^0 *is (zero state)* detectable *if* $w = 0$ *and* $\lim_{t\to\infty} z(t) = 0$ *implies* $\lim_{t\to\infty} x(t) = 0$.

The above-mentioned well known results are summarized in the next theorem and the remarks following it.

Theorem 3.2 *(See [30], [16], [28], [21], [5].) Assume that* Σ^0 *is dissipative with respect to the supply rate* $r(z, w)$ *in (2.2). Then there exist a non-negative solution* V, *with* $V(0) = 0$ *to the PDI (3.2) in the viscosity sense (see the Appendix for a definition). Conversely, if there exists a non-negative function* V, *with* $V(0) = 0$ *that satisfies the PDI in the viscosity sense, then* Σ^0 *is dissipative with respect to the supply rate* $r(z, w)$ *in (2.2), and the function* V_* *defined by*

$$V_*(x) = \liminf_{y\to x} V(y),$$

is a lower semicontinuous storage function.

Remark 3.3 Suppose that the supply rate $r(z, w)$ in (2.2) satisfies a stronger condition

$$r(z, 0) \leq -c_1|z|^2, \tag{3.3}$$

for all z, for some constant $c_1 > 0$. Assume that Σ^0 is dissipative with respect to $r(z, w)$. Then, by using inequality (2.4) with $w = 0$, and using (3.3) we get the inequality

$$0 \leq c_1 \int_0^T |h(x(t), 0)|^2 dt \leq -\int_0^T r(h(x(t), 0), 0)dt \ \leq \ \beta(x_0) < +\infty,$$

for all $T \geq 0$, for all initial conditions x_0 in $\mathbf{R}^n$. This implies that $\lim_{t\uparrow\infty} |h(x(t), 0)|^2 = 0$ and, therefore, $h(x(t), 0) \to 0$ as $t \uparrow \infty$. If the system Σ^0 is detectable then $x(t) \to 0$ as $t \uparrow \infty$. Thus, the equilibrium $x = 0$ is asymptotically stable. This provides a way of showing the asymptotic stability of dissipative systems under (zero-state) detectability assumption. Related results are presented in [16] (based on a Lyapunov function argument) and in [20].

Remark 3.4 Suppose that Σ^0 is linear, i.e., $A(x) = Ax$, $B_1(x) = B_1$, $C_1(x) = C_1x$, and $D_{11}(x) = D_{11}$, where A, B_1, C_1, D_{11} are constant matrices with appropriate sizes, and the supply rate has a quadratic form, i.e. $r(z, w) = \frac{1}{2}(w'Qw + 2w'Sz + z'Rz)$, for some constant matrices Q, S, R, with $Q = Q'$ and $R = R' \leq 0$, such that $\bar{Q} = Q + SD_{11} + D_{11}'S' + D_{11}'RD_{11} > 0$. Then the function V in Theorem 3.2 is quadratic given by $V(x) = \frac{1}{2}x'Xx$, where X is a non negative matrix satisfying the following ARI (see a related result in Theorem 3 of [31])

$$A'X + XA + [XB_1 - C_1'\bar{S}']\bar{Q}^{-1}[B_1'X - \bar{S}C_1] - C_1'RC_1 \leq 0, \tag{3.4}$$

where $\bar{S} = S + D_{11}R$. Arguing as in the previous remark, if $R < 0$ (which implies the condition (3.3) holds) and the matrices pair (A, C_1) is completely observable, then A is an asymptotically stable matrix, i.e., each eigenvalue of A has a negative real part.

3.2 Strict Dissipativity for Nonlinear Systems

In the paper [23], it is shown that if a linear system has a strict finite gain property and is asymptotically stable, then it posseses a (available) storage function with a stabilizability property. This result, which is expressed in terms of an algebraic Riccati inequality, was obtained without assuming controllability or observability of the system. We shall now present an extension of these results to a more general type of dissipativity for nonlinear systems.

We consider the nonlinear systems described in (3.1) and a general quadratic supply rate of the form

$$r_q(z,w) = \frac{1}{2}(w'Qw + 2w'Sz + z'Rz), \tag{3.5}$$

where Q, R are symmetric matrices. We assume $R \leq 0$ and $\bar{Q}(x) = Q + SD_{11}(x) + D_{11}'(x)S' + D_{11}'(x)RD_{11}(x) > 0$ for all x and write $\bar{S}(x) = S + D_{11}(x)'R$. This quadratic supply rate includes the following performances as a few special cases (see [17], [32]) :

(i) H_∞ (or finite gain) performance; when $Q = \gamma^2 I, S = 0, R = -I$,

(ii) positive real (or passivity) performance; when $Q = R = 0, S = I$,

(iii) mixed performance; when $Q = \theta\gamma^2 I, R = -\theta I, S = (1-\theta)I, \theta \in [0,1]$,

(iv) sector bounded performance; when $Q = -\frac{1}{2}(K_1'K_2 + K_2'K_1), S = \frac{1}{2}(K_1 + K_2)'$, $R = -I$, for some constant matrices K_1, K_2.

Before presenting our results, we shall make the following definitions.

Definition 3.5 *A vector field $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ is called* strongly stable *if there exists $c > 0$ such that*

$$(f(x) - f(y))'(x - y) \leq -c|x - y|^2,$$

for all x, y in $\mathbf{R}^n$. If f is globally asymptotically stable, it is called weakly stable.

The following theorem is a generalization of the strict bounded real lemma (SBRL) in [23].

Theorem 3.6 *Consider the open loop system Σ^0 (3.1). Suppose that the vector field $A(x)$ is strongly stable and Σ^0 is strictly dissipative with respect to the supply rate $r_q(z,w)$ in (3.5). Then:*

i. there exists a (locally bounded) lower semi continuous function $\tilde{V} > 0$, with $\tilde{V}(0) = 0$, satisfying the PDI

$$\sup_{w \in \mathbf{R}^d} \{\nabla_x \tilde{V}(x) \cdot (A(x) + B_1(x)w) - r(h(x,w),w) + \frac{\delta}{2}x'x\} \leq 0, \tag{3.6}$$

for some $\delta > 0$, where $h(x,w) = C_1(x) + D_{11}(x)w$, in the viscosity sense, and *ii. there exists a (locally bounded) function $V \geq 0$ with $V(0) = 0$, satisfying the PDE*

$$\sup_{w\in\mathbf{R}^d} \{\nabla_x V(x)\cdot(A(x) + B_1(x)w) - r(h(x,w),w)\} = 0, \tag{3.7}$$

in the viscosity sense. Moreover, if V is smooth, then the vector field

$$A^*(x) = A(x) + B_1(x)\bar{Q}^{-1}(x)[B_1(x)'\nabla_x V(x)' - \bar{S}(x)C_1(x)]$$

is weakly stable.

Conversely, suppose there exists a smooth solution $V \geq 0$, with $V(0) = 0$, to the PDE (3.7), in the classical sense such that the vector field A^* is strongly stable. Then the vector field A is weakly stable and Σ^0 is dissipative with respect to the supply rate $r_q(z,w)$ in (3.5). Moreover, if the vector field $w^*(x) = \bar{Q}^{-1}(x)[B_1(x)'\nabla_x V(x)' - \bar{S}(x)C_1(x)]$ satisfies $|w^*(x)| \leq \alpha|x|$, for all x, for some $\alpha > 0$, then the dissipativity is strict.

PROOF. For any $w \in L_2([0,T],\mathbf{R}^d)$ and any $T \geq 0$, the strict dissipativity of Σ^0 implies

$$-\int_0^T r_q(z(t),w(t))dt \;\leq\; -\frac{\epsilon}{2}\int_0^T w'(t)w(t)dt + \beta(x_0), \text{ for some } \epsilon > 0,$$

and the strong stability of A implies (see [9])

$$\frac{1}{2}\int_0^T x'(t)x(t)dt \;\leq\; \frac{\bar{\gamma}}{2}\int_0^T w'(t)w(t)dt + \bar{\beta}(x_0), \text{ for some } \bar{\gamma} > 0,$$

where β, $\bar{\beta}$, ϵ, $\bar{\gamma}$ are independent of T, w. Multiplying both sides of the last inequality by $\epsilon/\bar{\gamma}$, and adding the result to the first one yields, for any $w \in L_2([0,T],\mathbf{R}^d)$, $T \geq 0$,

$$-\int_0^T (r_q(z(t),w(t)) - \frac{\delta}{2}x'(t)x(t))dt \;\leq\; \beta(x_0) + \delta\bar{\beta}(x_0) = \tilde{\beta}(x_0),$$

where $\delta = \epsilon/\bar{\gamma} > 0$. This implies that the system $\hat{\Sigma}^0$ defined by

$$\hat{\Sigma}^0 : \begin{cases} \dot{x}(t) &= A(x(t)) + B_1(x(t))w(t),\; x(0) = x_0, \\ \hat{z}(t) &= \begin{bmatrix} C_1(x(t)) \\ \delta^{\frac{1}{2}}x(t) \end{bmatrix} + \begin{bmatrix} D_{11}(x(t)) \\ 0 \end{bmatrix} w(t), \end{cases}$$

is dissipative with respect to the supply rate

$$\hat{r}_q(\hat{z},w) = \frac{1}{2}(w'Qw + 2w'\hat{S}\hat{z} + \hat{z}'\hat{R}\hat{z}),$$

where $\hat{S} = [S \;\; 0]$, $\hat{R} = \mathrm{diag}(R,-I)$. Write $\hat{h}(x,w) = \begin{bmatrix} C_1(x(t)) \\ \delta^{\frac{1}{2}}x(t) \end{bmatrix} + \begin{bmatrix} D_{11}(x(t)) \\ 0 \end{bmatrix} w(t)$. Since

$$\hat{r}_q(\hat{h}(x,w),w) = \frac{1}{2}(w'Qw + 2w'Sh(x,w) + h(x,w)'Rh(x,w) - \delta x'x),$$

Theorem 3.1 of [21] implies there exists a lower semicontinuous storage function $\hat{V} \geq 0$, with $\hat{V}(0) = 0$, solving the PDI

$$\begin{aligned} &\sup_{w\in\mathbf{R}^d}\{\nabla_x \hat{V}(x)\cdot(A(x)+B_1(x)w) - \tfrac{1}{2}(w'Qw + 2w'Sh(x,w) \\ &\quad + h(x,w)'Rh(x,w) - \delta x'x)\} \\ &\qquad\qquad = \sup_{w\in\mathbf{R}^d}\{\nabla_x \hat{V}(x)\cdot(A(x)+B_1(x)w) - \hat{r}_q(\hat{h}(x,w),w)\} \\ &\qquad\qquad \leq 0. \end{aligned} \tag{3.8}$$

This proves the existence of the PDI (3.6). The function $\hat{V}$ has the integral representation

$$\begin{aligned} \hat{V}(x) \;\geq\; &\sup_{T\geq 0, w\in L_2([0,T],\mathbf{R}^d)}\{\hat{V}(x(T)) - \tfrac{1}{2}\int_0^T (w(t)'Qw(t) + 2w(t)'Sh(x(t),w(t)) \\ &+ h(x(t),w(t))'Rh(x(t),w(t)) - \delta x(t)'x(t))dt : x(0) = x\}. \end{aligned} \tag{3.9}$$

Suppose $\hat{V}(x) = 0$, for some x. Then, (3.9) implies, with $T \geq 0,\ w = 0$

$$0 = \hat{V}(x) \geq \delta \frac{1}{2}\int_0^T x'(t)x(t)dt \geq 0.$$

Thus $x(t) = 0$ for all $t \in [0,\ T]$. Uniqueness of solutions to ODE then implies that $x = x(0) = 0$. This proves assertion (i).

Next, define the function V by

$$\begin{aligned} V(x) = &\sup_{T\geq 0, w\in L_2([0,T],\mathbf{R}^d)}\{-\tfrac{1}{2}\int_0^T (w(t)'Qw(t) + 2w(t)'Sh(x(t),w(t)) \\ &+ h(x(t),w(t))'Rh(x(t),w(t)))dt : x(0) = x\}. \end{aligned}$$

Clearly we have $0 \leq V \leq \hat{V}$. By Theorem 3.1 of [21], its lower semicontinuous envelope V_* is a viscosity supersolution of the PDE (3.7). Moreover, its upper envelope V^* is a viscosity subsolution of the PDE (3.7) (see Appendix B for a proof, see also [5], [25]). Thus, we conclude that V solves the PDE (3.7) in the viscosity sense.

Next, suppose V is smooth, and set $w = w^*(x) = \bar{Q}^{-1}(x)[B_1'(x)\nabla_x V(x)' - \bar{S}(x)C_1(x)]$ (which achieves the maximum in the PDE (3.7)). Then we get

$$\begin{aligned} V(x(0)) - V(x(T)) = -\tfrac{1}{2}\int_0^T &(w^*(t)'Qw^*(t) + 2w^*(t)'Sh(x(t),w^*(t)) \\ &+ h(x(t),w^*(t))'Rh(x(t),w^*(t)))dt. \end{aligned} \tag{3.10}$$

The representation (3.9) implies that for $w = w^*$

$$\hat{V}(x(0)) - \hat{V}(x(T)) \geq -\tfrac{1}{2}\int_0^T (w^*(t)'Qw^*(t) + 2w^*(t)'Sh(x(t), w^*(t)) \\ + h(x(t), w^*(t))'Rh(x(t), w^*(t)) - \delta x(t)'x(t))dt. \tag{3.11}$$

Subtracting (3.10) from (3.11) we get

$$S(x(0)) - S(x(T)) \geq \frac{1}{2}\delta \int_0^T x(t)'x(t)dt,$$

where $S = \hat{V} - V \geq 0$, with $S(0) = 0$. Since the last inequality implies

$$S(x(0)) \geq \frac{1}{2}\delta \int_0^T x(t)'x(t)dt,$$

for all $T \geq 0$ we conclude that any initial $x(0)$ produces a $L_2([0,\infty), \mathbf{R}^n)$ trajectory $x(\cdot)$; therefore $x(t) \to 0$. This shows the weak stability of the vector field A^*.

To prove the converse, write

$$\dot{x} = A(x),\ x(0) = x_0,$$

$$\dot{\xi} = A^*(\xi),\ \xi(0) = x_0.$$

The PDE (3.7) implies (since $-R \geq 0$)

$$\nabla_x V(x) \cdot A(x) + \tfrac{1}{2}[\nabla_x V(x) \cdot B_1(x) - C_1(x)'\bar{S}(x)']\bar{Q}^{-1}(x)\bar{Q}(x) \\ \times \bar{Q}^{-1}[B_1(x)' \cdot \nabla_x V(x)' - \bar{S}(x)C_1(x)] \leq 0,$$

therefore

$$V(x(0)) \geq \tfrac{1}{2}K\int_0^T ([\nabla_x V(x(t)) \cdot B_1(x(t)) - C_1(x(t))'\bar{S}(x(t))']\bar{Q}^{-1}(x(t))) \\ (\bar{Q}^{-1}(x(t))[B_1(x(t))' \cdot \nabla_x V(x(t))' - \bar{S}(x(t))C_1(x(t))])dt,$$

with $K = \|\sigma_{min}(\bar{Q}(x))\|_{L_\infty}$, which shows that the trajectory of $w^*(t) = w^*(x(t)) = \bar{Q}^{-1}(x(t))[B_1(x(t))' \cdot \nabla_x V(x(t))' - \bar{S}(x(t))C_1(x(t))]$ is in $L_2([0,\infty), \mathbf{R}^d)$. Now, define $e = \xi - x$. Then, by the hypothesis that $A^*(x)$ is strongly stable, and using a technique similar to that in [9], one can obtain the following estimate for $e(t)$

$$\tfrac{1}{2}e'(t)e(t) \leq \exp(-\bar{c}t)e'(0)e(0) + \tfrac{1}{2}\bar{K}\int_0^t \exp(-\bar{c}(t-s))|\bar{Q}^{-1}(x(s))[B_1(x(s))' \cdot \nabla_x V(x(s))' \\ -\bar{S}(x(s))C_1(x(s))]|^2 ds,$$

for some constants $\bar{c} > 0, \bar{K} > 0$. Therefore $e(\cdot)$ is in $L_2([0,\infty), \mathbf{R}^n)$. In particular, $e(t) \to 0$. Since $\xi(t) \to 0$, we conclude that $x(\cdot)$ is asymptotically stable, and thus A is weakly stable. Next, by direct calculation we get

$$
\begin{aligned}
&-\int_0^T r_q(z(s), w(s))ds \\
&\qquad = -\tfrac{1}{2}\int_0^T ([w(s) - w^*(s)]'\bar{Q}[w(s) - w^*(s)] + 2dV/ds)ds \\
&\qquad = -\tfrac{1}{2}\int_0^T [w(s) - w^*(s)]'\bar{Q}[w(s) - w^*(s)]ds + V(x_0) - V(x(T)), \\
&\qquad \leq -\tfrac{1}{2}\|\sigma_{min}(\bar{Q})(x)\|_{L_\infty}\int_0^T [w(s) - w^*(s)]'[w(s) - w^*(s)]ds + V(x_0),
\end{aligned}
\tag{3.12}
$$

since $V \geq 0$, where $w^*(s) = w^*(x(s)) = \bar{Q}^{-1}(x(s))[B_1(x(s))'\nabla_x V(x(s))' - \bar{S}(x(s))C_1(x(s))]$. The last inequality shows that Σ^0 is dissipative with respect to the supply rate $r_q(z, w)$ (3.5), with $\beta = V$. To show the strict dissipativity, consider the following system

$$
\Sigma^{-1} : \left\{ \begin{array}{rcl} \dot{x}(t) &=& A^*(x(t)) + B_1(x(t))(w(t) - w^*(t)),\ x(0) = x_0, \\ \\ w(t) &=& w^*(t) + (w(t) - w^*(t)). \end{array} \right.
$$

Since $w,\ w^*$ are in $L_2([0,\infty), \mathbf{R}^d)$ so is $w - w^*$. Since A^* is strongly stable, we have (see [9])

$$
\frac{1}{2}\int_0^T x'(t)x(t)dt \ \leq\ \frac{1}{2}\gamma^*\int_0^T (w(t) - w^*(t))'(w(t) - w^*(t))dt + \beta^*(x_0), \tag{3.13}
$$

for all $T \geq 0$, for all $w - w^* \in L_2([0,T], \mathbf{R}^d)$, for some $\beta^* \geq 0$ with $\beta^*(0) = 0$, for some $\gamma^* > 0$. Furthermore, by using $w = w^* + (w - w^*)$, the inequality (3.13) and the hypothesis $|w^*(x)| \leq \alpha|x|$, for all x, we have

$$
\begin{aligned}
\tfrac{1}{2}\int_0^T w'(t)w(t)dt \ &\leq\ \frac{1}{2}\int_0^T w^*(t)'w^*(t)dt + \frac{1}{2}\int_0^T (w(t) - w^*(t))'(w(t) - w^*(t))dt \\
&\leq\ \tfrac{1}{2}K_1\int_0^T (w(t) - w^*(t))'(w(t) - w^*(t))dt + \beta_1(x_0),
\end{aligned}
\tag{3.14}
$$

where $K_1 = \alpha\gamma^* + 1,\ \beta_1 = \alpha\beta^*$. By substituting (3.14) in (3.12) we finally get

$$
\begin{aligned}
&-\int_0^T r_q(z(s), w(s))ds \\
&\qquad \leq -\tfrac{1}{2}(\|\sigma_{min}(\bar{Q})(x)\|_{L_\infty}/K_1)\int_0^T w(s)'w(s)ds + V(x_0) + \beta_1(x_0)/K_1,
\end{aligned}
$$

for all $w \in L_2([0,T],\mathbf{R}^d)$, for all $T \geq 0$, which shows the required strict dissipativity with $\epsilon = \|\sigma_{min}(\bar{Q})(x)\|_{L_\infty}/K_1 > 0$, and $\beta(x) = V(x) + \beta_1(x)/K_1$. This completes the proof.

$\square$

The following corollary shows the connection between the dissipativity of the system Σ^0 (3.1) and the finite gain property of a related system $\tilde{\Sigma}^0$ given by

$$\tilde{\Sigma}^0 : \begin{cases} \dot{x}(t) &= \tilde{A}(x(t)) + \tilde{B}_1(x(t))w(t), \; x(0) = x_0, \\ \\ z(t) &= \tilde{C}_1(x(t)), \end{cases} \tag{3.15}$$

where $\tilde{A}(x) = A(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)C_1(x)$, $\tilde{B}_1(x) = B_1(x)\bar{Q}^{-\frac{1}{2}}(x)$, and $\tilde{C}_1(x) = (\bar{S}(x)'\bar{Q}^{-1}(x)\bar{S}(x) - R)^{\frac{1}{2}}C_1(x)$.

Corollary 3.7 *Consider the system Σ^0 and the quadratic supply rate $r_q(z,w)$ in (3.5). Assume that the vector field $A(x)$ is strongly stable and the system Σ^0 is strictly dissipative with respect to the supply rate $r_q(z,w)$ in (3.5). Then, there exists a solution $V \geq 0$, with $V(0) = 0$, to the PDI*

$$\nabla_x V(x) \cdot \tilde{A}(x) + \frac{1}{2}\nabla_x V(x) \cdot \tilde{B}_1(x)\tilde{B}_1(x)' \cdot \nabla_x V(x)' + \frac{1}{2}\tilde{C}_1(x)'\tilde{C}_1(x) \leq 0 \tag{3.16}$$

in the viscosity sense. In particular, the system $\tilde{\Sigma}^0$ has finite gain less than 1. If, moreover V is smooth, then the vectorfield $\tilde{A}(x) + \tilde{B}_1(x)\tilde{B}_1(x)'\nabla_x V(x)'$ is weakly stable.

Conversely, assume that the vector field $\tilde{A}(x) = A(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)C_1(x)$ is strongly stable and the system $\tilde{\Sigma}^0$ has finite gain strictly less than 1. Then there exists a solution $V \geq 0$, with $V(0) = 0$, to the PDI (3.7) in the viscosity sense. In particular, the system Σ^0 is dissipative with respect to the supply rate $r_q(z,w)$ in (3.5). If V is smooth, then $A^*(x) = A(x) + B_1(x)\bar{Q}^{-1}(x)[B_1(x)'\nabla_x V(x)' - \bar{S}(x)C_1(x)]$ is weakly stable.

PROOF. Under the hypothesis, Theorem 3.6 implies there exists a solution $V \geq 0$ to the PDI (3.7) in the viscosity sense. This PDI can be rewritten as in (3.16). Therefore, it admits the following integral representation (see Theorem 3.1 of [21])

$$\begin{aligned} V(x) \;\geq\; & \sup_{T\geq 0, w\in L_2([0,T],\mathbf{R}^d)}\{V(x(T)) \\ & - \tfrac{1}{2}\int_0^T (w(t)'w(t) - \tilde{C}_1'(x(t))\tilde{C}_1(x(t)))dt : x(0) = x\}, \end{aligned}$$

where $x(\cdot)$ is the trajectory generated by the system $\tilde{\Sigma}^0$ in (3.15). This implies that $\tilde{\Sigma}^0$ has finite gain less than 1. If V is smooth, then results in part (ii) of Theorem 3.6 implies that the vector field

$$\begin{aligned} \tilde{A}(x) + \tilde{B}_1(x)\tilde{B}_1(x)'\nabla_x V(x)' = \;& A(x) + B_1(x)\bar{Q}^{-1}(x)[B_1(x)'\nabla_x V(x)' - \bar{S}(x)C_1(x)] \\ = \;& A^*(x) \end{aligned}$$

is weakly stable.

Conversely, if the system $\tilde{\Sigma}^0$ has finite gain strictly less than 1 with $\tilde{A}$ strongly stable, then Theorem 3.6 implies there exists a solution to the PDI (3.16) in the viscosity sense. Rewriting this PDI as in (3.7) and applying Theorem 3.1 of [21] results in the representation

$$\begin{aligned} V(x) \ \geq \ & \sup_{T\geq 0, w\in L_2([0,T],\mathbf{R}^d)}\{V(x(T)) - \tfrac{1}{2}\int_0^T (w(t)'Qw(t) + 2w(t)'Sh(x(t),w(t)) \\ & + h(x(t),w(t))'Rh(x(t),w(t)))dt : x(0) = x\}, \end{aligned}$$

where $x(\cdot)$ is the trajectory produced by Σ^0 in (3.1). Thus the system Σ^0 is dissipative with respect to the supply rate $r_q(z,w)$ (3.5). Moreover, if V is smooth, then by Theorem 3.6 the vector field

$$\begin{aligned} A^*(x) = \ & A(x) + B_1(x)\bar{Q}^{-1}(x)[B_1(x)'\nabla_x V(x)' - \bar{S}(x)C_1(x)] \\ = \ & \tilde{A}(x) + \tilde{B}_1(x)\tilde{B}_1(x)'\nabla_x V(x)' \end{aligned}$$

is weakly stable. This completes the proof.

□

3.3 Strict Dissipativity for Linear Systems

In this section we consider linear systems described by

$$\Sigma_l^0 : \begin{cases} \dot{x}(t) \ = \ Ax(t) + B_1 w(t), \ x(0) = x_0, \\ z(t) \ = \ C_1 x(t) + D_{11} w(t), \end{cases} \tag{3.17}$$

where A, B_1, C_1, D_{11} are appropriately sized constant matrices. The supply rate we consider has a quadratic form

$$r_q(z,w) = \frac{1}{2}(w'Qw + 2w'Sz + z'Rz), \tag{3.18}$$

where Q, R are symmetric matrices. We assume $R \leq 0$ and $\bar{Q} = Q + SD_{11} + D_{11}'S' + D_{11}'RD_{11} > 0$ and write $\bar{S} = S + D_{11}'R$. The strong and weak stability concepts can both be replaced by the Hurwitz stability criterion. We shall now present the analogy of Theorem 3.6 for linear systems (see also the bounded real lemmas in [23], [13] and a more general dissipativity case in [31]).

Theorem 3.8 *Assume that the matrix A is asymptotically stable and $\bar{Q} = Q + SD_{11} + D_{11}'S' + D_{11}'RD_{11} > 0$, $R \leq 0$. The linear system Σ_l^0 in (3.17) is strictly dissipative with* *respect to the supply rate $r_q(z,w)$ in (3.18) if and only if there exists a solution $X \geq 0$ to the following ARE*

$$A'X + XA + [XB_1 - C_1'\bar{S}']\bar{Q}^{-1}[B_1'X - \bar{S}C_1] - C_1'RC_1 = 0, \tag{3.19}$$

such that the matrix $A^* = A + B\bar{Q}^{-1}[B_1'X - \bar{S}C_1]$ is asymptotically stable.

PROOF. We follow the technique used in the proof of Theorem 3.7.1 in [13].

(Necessity) First note that the strict dissipativity implies that with $x(0) = 0$ we have for any $T \geq 0$

$$-\int_0^T r_q(z(t), w(t))dt \;\leq\; -\frac{1}{2}\epsilon\int_0^T w'(t)w(t)dt, \tag{3.20}$$

for all $w \in L_2([0,T], \mathbf{R}^d)$. We shall show that this implies that for any $T \geq 0$, the following Riccati differential equation (RDE)

$$-\dot{\tilde{X}} = A'\tilde{X}(t) + \tilde{X}(t)A + [\tilde{X}(t)B_1 - C_1'\bar{S}']\bar{Q}^{-1}[B_1'\tilde{X}(t) - \bar{S}C_1] - C_1'RC_1, \tag{3.21}$$

with $\tilde{X}(T,T) = 0$, has no finite escape time in $[0,T]$, and moreover, $\tilde{X}(t,T) \geq 0$. Having done this, we shall show that the limit $0 \leq X = \lim_{T-t\uparrow\infty} \tilde{X}(t,T)$ is the required stabilizing solution to the ARE (3.19).

Consider the following two point boundary value problem (TPBVP)

$$\begin{bmatrix} \dot{x}(t) \\ \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A - B_1\bar{Q}^{-1}\bar{S}C_1 & B_1\bar{Q}^{-1}B_1' \\ \\ -C_1'(\bar{S}'\bar{Q}^{-1}\bar{S} - R)C_1 & -(A - B_1\bar{Q}^{-1}\bar{S})' \end{bmatrix} \begin{bmatrix} x(t) \\ \\ p(t) \end{bmatrix}, \quad \begin{bmatrix} x(t_0) \\ \\ p(T) \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}, \tag{3.22}$$

where $0 \leq t_0 \leq T$. We shall show that this TPBVP has no nontrivial solution on $[t_0, T]$, for all $0 \leq t_0 \leq T$. The result follows trivially if $t_0 = T$. Thus, we assume $t_0 < T$. Consider any solution $x(\cdot), p(\cdot)$ to the TPBVP and set $w = \tilde{w}$, where

$$\tilde{w}(t) \;=\; \begin{cases} \bar{Q}^{-1}(B_1'p(t) - \bar{S}C_1x(t)), & t_0 < t \leq T, \\ \\ 0, & 0 \leq t \leq t_0. \end{cases}$$

Note that applying $\tilde{w}$ on Σ_l^0 in (3.17) with $x(0) = 0$ yields $x(t) = 0, z(t) = 0,\ t \in [0, t_0]$. By direct calculation, we get

$$\begin{aligned} -\int_0^T r_q(z(t), \tilde{w}(t))dt &= -\int_{t_0}^T r_q(z(t), \tilde{w}(t))dt \\ &= -\tfrac{1}{2}\int_{t_0}^T (p(t)'B_1\bar{Q}^{-1}B_1'p(t) - x'(t)C_1'(\bar{S}'\bar{Q}^{-1}\bar{S} - R)C_1x(t))dt \\ &= -\tfrac{1}{2}\int_{t_0}^T d(x(t)'p(t))/dt dt \\ &= -\tfrac{1}{2}(x(T)'p(T) - x(t_0)'p(t_0)) \\ &= 0. \end{aligned}$$

Thus, we have (using (3.20)) $0 = -\int_{t_0}^{T} r_q(z(t), \tilde{w}(t))dt \leq -\frac{1}{2}\epsilon\int_{t_0}^{T} \tilde{w}(t)'\tilde{w}(t)dt \leq 0$, from which we conclude that $\tilde{w}(\cdot) = 0$ on $[t_0, T]$. Since $\bar{Q} > 0$ we must have $B_1 p(\cdot) = \bar{S}C_1 x(\cdot)$ on $[t_0, T]$. The TPBVP now becomes

$$\begin{bmatrix} \dot{x}(t) \\ \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ \\ C_1 R C_1 & -A' \end{bmatrix} \begin{bmatrix} x(t) \\ \\ p(t) \end{bmatrix}, \quad \begin{bmatrix} x(t_0) \\ \\ p(T) \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix},$$

which implies $x(\cdot) = 0,\ p(\cdot) = 0$ on $[t_0, T]$. Thus, we conclude that the TPBVP has only trivial solutions on $[t_0, T]$ for all $t_0 \in [0, T]$. Next, let $\Phi(t, T)$ denote the transition matrix associated with the TPBVP (3.22), i.e.

$$\begin{bmatrix} \dot{\Phi}_{11}(t,T) & \dot{\Phi}_{12}(t,T) \\ \\ \dot{\Phi}_{21}(t,T) & \dot{\Phi}_{22}(t,T) \end{bmatrix} = \begin{bmatrix} A - B_1\bar{Q}^{-1}\bar{S}C_1 & B_1\bar{Q}^{-1}B_1' \\ \\ -C_1'(\bar{S}'\bar{Q}^{-1}\bar{S} - R)C_1 & -(A - B_1\bar{Q}^{-1}\bar{S})' \end{bmatrix}$$

$$\times \begin{bmatrix} \Phi_{11}(t,T) & \Phi_{12}(t,T) \\ \\ \Phi_{21}(t,T) & \Phi_{22}(t,T) \end{bmatrix},$$

$\Phi(T, T) = I$. The fact that the TPBVP has only trivial solution implies that $\Phi_{11}(t, T)$ is non-singular for all $0 \leq t \leq T$. To see this, pick any vector v such that $\Phi_{11}(t_0, T)v = 0$ for some $t_0 \in [0, T]$. Setting $x(T) = v,\ p(T) = 0$ we get

$$\begin{bmatrix} x(t_0) \\ p(t_0) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t_0,T) \\ \Phi_{21}(t_0,T) \end{bmatrix} v = \begin{bmatrix} 0 \\ \Phi_{21}(t_0,T) \end{bmatrix}.$$

By the previous results, we must have $x(\cdot) = 0$ and $p(\cdot) = 0$ for all $t_0 \leq t \leq T$. In particular $p(t_0) = 0$. Thus, we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t_0,T) \\ \Phi_{21}(t_0,T) \end{bmatrix} v = \Phi(t,T) \begin{bmatrix} v \\ 0 \end{bmatrix}.$$

Since the transition matrix $\Phi(t, T)$ is nonsingular for all $t \in [0, T]$, we must have $v = 0$. This shows that $\Phi_{11}(t_0, T)$ is nonsingular for all $t_0 \in [0, T]$. Next, it is straightforward to see that the matrix $\tilde{X}(t, T)$ defined by

$$\tilde{X}(t,T) = \Phi_{21}(t,T)\Phi_{11}(t,T)^{-1}$$

solves the following RDE

$$-\dot{\tilde{X}} = A'\tilde{X} + \tilde{X}A + [\tilde{X}B_1 - C_1'\bar{S}']\bar{Q}^{-1}[B_1'\tilde{X} - \bar{S}C_1] - C_1'RC_1, \qquad (3.23)$$

with terminal condition $\tilde{X}(T, T) = 0$. To show that $\tilde{X}(t, T) \geq 0$ for all $t \in [0, T]$, define the quadratic function $\tilde{V}(x, t) = \frac{1}{2}x'\tilde{X}(t, T)x$. This function has the integral representation

$$\tilde{V}(x,t) = \sup_{w \in L_2([0,T],\mathbf{R}^d)}\{-\tfrac{1}{2}\int_t^T (w(s)'Qw(s) + 2w(s)'Sh(x(s), w(s))$$

$$+\ h(x(s), w(s))'Rh(x(s), w(s))ds : x(t) = x\}.$$

Setting $w = 0$ we get

$$\tilde{V}(x,t) \geq \{-\tfrac{1}{2}\int_t^T h(x(s),0)'Rh(x(s),0)ds : x(t) = x\}$$

$$\geq 0,$$

for all $x \in \mathbf{R}^n$, for all $t \in [0,T]$, since $R \leq 0$. Thus, we conclude that $\tilde{X}(t,T) \geq 0$ for all $t \in [0,T]$.

Next, we shall show that the limit

$$X = \lim_{T-t\uparrow\infty} \tilde{X}(t,T) \tag{3.24}$$

exists. Differentiating both sides of the RDE (3.23) yields the matrix differential equation

$$-\ddot{\tilde{X}} = (A - B_1\bar{Q}^{-1}\bar{S}C_1)'\dot{\tilde{X}} + \dot{\tilde{X}}(A - B_1\bar{Q}^{-1}\bar{S}C_1),$$

with $-\dot{\tilde{X}}(T,T) = C_1'(\bar{S}'\bar{Q}^{-1}\bar{S} - R)C_1 \geq 0$. The solution to this equation is given by

$$\dot{\tilde{X}}(t,T) = -\tilde{\Phi}(t,T)C_1'(\bar{S}'\bar{Q}^{-1}\bar{S} - R)C_1\tilde{\Phi}'(t,T),$$

where $\tilde{\Phi}(t,T)$ is the transition matrix associated with $(A - B_1\bar{Q}^{-1}\bar{S}C_1)$. This shows that $\tilde{X}(t,T)$ is monotonically nonincreasing in t. Furthermore, since the matrix A is asymptotically stable, the zero-input response (i.e. the response obtained by setting $w = 0$) z_{x_0} of Σ_l^0 satisfies

$$\int_0^T z_{x_0}(t)'z_{x_0}(t)dt \leq \alpha x_0'x_0,$$

for all $T \geq 0$, for all x_0, for some $\alpha > 0$. By using this inequality and (3.20), and employing the Cauchy-Schwarz inequality one can get the following upper bound for $\tilde{X}(t,T)$

$$\tilde{X}(t,T) \leq c,$$

for some constant $c > 0$ which is independent of t, T and x_0. Now, since $0 \leq \tilde{X}(t,T) \leq c$ and $\tilde{X}(t,T)$ is monotonic, the limit X in (3.24) exists and is nonnegative. Moreover, because the solution of the RDE (3.23) depends continuously upon its terminal condition, one can conclude that the limit $X = \lim_{T-t\uparrow\infty} \tilde{X}(t,T)$ solves the ARE in (3.19).

We shall now show that the matrix $A^* = A + B\bar{Q}^{-1}[B_1'X - \bar{S}C_1]$ is asymptotically stable. First, we note that by the Plancherel's theorem the strict dissipation inequality (3.20) implies

$$-\frac{1}{2}\int_{-\infty}^{\infty} \hat{w}(j\omega)^*H(j\omega)\hat{w}(j\omega)d\omega \leq -\frac{\epsilon}{2}\int_{-\infty}^{\infty} \hat{w}(j\omega)^*\hat{w}(j\omega)d\omega$$

for all $\hat{w} \in L_2((-\infty,\infty),\mathbf{R}^d)$, where $\hat{w}$ denotes the Fourier transform of w and $H(j\omega)$ is given by

$$H(j\omega) = Q + SG(j\omega) + G(-j\omega)'S' + G(-j\omega)'RG(j\omega), \tag{3.25}$$

where $G(j\omega) = D_{11} + C_1(j\omega I - A)^{-1}B_1$. This inequality implies (as can be shown by using a simple contradictive argument) that $H(j\omega)$ satisfies the positive definiteness condition

$$H(j\omega) > \eta I, \tag{3.26}$$

for all $\omega \in \mathbf{R}$, for some constant $\eta > 0$. Next, define the transfer matrix $\mathbf{W}(j\omega) = W + WL(j\omega I - A)^{-1}B$, where $W'W = \bar{Q} > 0$, $L = -\bar{Q}^{-1}[B_1'X - \bar{S}C_1]$. Then, we obtain the following factorization

$$H(j\omega) = \mathbf{W}(-j\omega)'\mathbf{W}(j\omega).$$

The condition (3.26) implies that $\mathbf{W}(j\omega)$ has no purely imaginary zeros. Moreover, the stability of the matrix A guarantees there is no unobservable/uncontrollable modes on the $j\omega$ axis. Noting that $\mathbf{W}^{-1}(j\omega) = W^{-1} - L(j\omega I - (A - B_1 L))^{-1}B_1 W^{-1}$, with $A - B_1 L = A + B_1\bar{Q}^{-1}[B_1'X - \bar{S}C_1] = A^*$, we conclude that the matrix A^* has no purely imaginary eigenvalues. Finally, using a technique similar to that in the proof of Lemma 3.7.7 of [13], one can show that the existence of the limit $X \geq 0$ given in (3.24), satisfying the ARE (3.19) implies that the matrix $A^* = A + B_1\bar{Q}^{-1}[B_1'X - \bar{S}C_1]$ has no eigenvalues with positive real parts. Combining these results, we conclude that the matrix A^* is asymptotically stable.

(Sufficiency) The ARE may be rewritten as

$$A'X + XA + \tilde{C}'\tilde{C} = 0,$$

where $\tilde{C} = \begin{bmatrix} (\bar{Q}^{-\frac{1}{2}}[B_1'X - \bar{S}C_1]) \\ \\ ((-R)^{\frac{1}{2}}C_1) \end{bmatrix}$. Since $A^* = A + [[XB_1 - C_1'\bar{S}']\bar{Q}^{-\frac{1}{2}} \;\; 0]\tilde{C}$ is asymptotically stable, the pair $(A, \tilde{C})$ is detectable, and since $X \geq 0$ solves the above (linear) Lyapunov equation, we conclude that A is asymptotically stable.

Next, for any output trajectory $z(\cdot)$ produced by Σ_l^0 in (3.17) with $x(0) = x_0$ we have

$$\begin{aligned} &-\int_0^T r_q(z(s), w(s))ds \\ &\quad = -\tfrac{1}{2}\int_0^T [w(s) - w^*(s)]'\bar{Q}[w(s) - w^*(s)]ds + \frac{1}{2}(x_0'Xx_0 - x'(T)Xx(T)) \qquad (3.27) \\ &\quad \leq -\tfrac{1}{2}\sigma_{min}(\bar{Q})\int_0^T [w(s) - w^*(s)]'[w(s) - w^*(s)]ds + \frac{1}{2}x_0'Xx_0, \end{aligned}$$

where $w^* = \bar{Q}^{-1}[B1'X' - \bar{S}C_1]$, for all $w \in L_2([0, T], \mathbf{R}^d)$ for all $T \geq 0$. This shows the dissipativity of Σ_l^0 with respect to $r_q(z, w)$. To get the strict dissipativity, consider the following system

$$\Sigma_l^{-1} : \left\{ \begin{array}{rcl} \dot{x}(t) & = & A^*x(t) + B_1\hat{w}(t), \;\; x(0) = x_0, \\ \\ w(t) & = & w^*(t) + \hat{w}(t), \end{array} \right. \tag{3.28}$$

where $\hat{w}(t) = w(t) - w^*(t)$. The output w can be written as $w = w_{x_0} + w_{\hat{w}}$, where w_{x_0} and $w_{\hat{w}}$ denote the zero-input and the zero-initial responses respectively. Since, by the hypothesis, A^* is aymptotically stable, the map $\Sigma_l^{-1} : \hat{w} \to w$, with initial condition $x_0 = 0$, has finite L_2 gain, i.e.,

$$\frac{1}{2}\int_0^T w'_{\hat{w}}(t)w_{\hat{w}}(t)dt \ \leq \ \frac{1}{2}\gamma^* \int_0^T |\hat{w}(t)|^2 dt, \tag{3.29}$$

for all $T \geq 0$, for all $\hat{w} \in L_2([0,T], \mathbf{R}^d)$, for some $\gamma^* > 0$. Moreover, by setting $\hat{w} = 0$ in (3.28), the stability of A^* implies

$$\frac{1}{2}\int_0^T w_{x_0}(t)'w_{x_0}(t)^* dt \ \leq \ \frac{1}{2}\alpha^* x_0' x_0, \tag{3.30}$$

for all $T \geq 0$, for all $x_0 \in \mathbf{R}^n$, for some $\alpha^* > 0$. Combining (3.29) and (3.30) and using the triangular inequality we get,

$$\begin{aligned} \tfrac{1}{2}\int_0^T w'(t)w(t)dt \ &\leq \ \frac{1}{2}\int_0^T w_{x_0}(t)'w_{x_0}(t)dt + \frac{1}{2}\int_0^T w'_{\hat{w}}w_{\hat{w}}(t)dt \\ &\leq \ \alpha^* x_0' x_0 + \gamma^* \int_0^T (w(t) - w^*(t))'(w(t) - w^*(t))dt. \end{aligned} \tag{3.31}$$

By substituting (3.31) in (3.27) we finally get

$$\begin{aligned} &-\int_0^T r_q(z(s), w(s))ds \\ &\qquad \leq -\tfrac{1}{2}\tfrac{\sigma_{min}(\bar{Q})}{\gamma^*}\int_0^T w(s)'w(s)ds + \frac{1}{2}x_0'(\frac{\alpha^*}{\gamma^*}I + X)x_0, \end{aligned}$$

for all $w \in L_2([0,T], \mathbf{R}^d)$, for all $T \geq 0$, which shows the required strict dissipativity with $\epsilon = \sigma_{min}(\bar{Q})/\gamma^* > 0$, and $\beta(x) = \frac{1}{2}x'(\frac{\alpha^*}{\gamma^*}I + X)x$. This completes the proof.

□

We shall now present the analogy of Corollary 3.7 for linear systems. Consider the following system $\tilde{\Sigma}_l^0$ which is related to the system Σ_l^0 in (3.17)

$$\tilde{\Sigma}_l^0 : \begin{cases} \dot{x}(t) \ = \ \tilde{A}x(t) + \tilde{B}_1 w(t), \ x(0) = x_0, \\ \\ z(t) \ = \ \tilde{C}_1 x(t), \end{cases} \tag{3.32}$$

where $\tilde{A} = A - B_1\bar{Q}^{-1}\bar{S}C_1$, $\tilde{B}_1 = B_1\bar{Q}^{-\frac{1}{2}}$, and $\tilde{C}_1 = (\bar{S}'\bar{Q}^{-1}\bar{S} - R)^{\frac{1}{2}}C_1$.

Corollary 3.9 *Assume that the matrix A is asymptotically stable and $\bar{Q} = Q + SD_{11} + D_{11}'S' + D_{11}'RD_{11} > 0$, $R \leq 0$. Suppose that the linear system Σ_l^0 (3.17) is strictly* *dissipative with respect to the supply rate $r_q(z,w)$ (3.18). Then, there exist a solution $X \geq 0$ to the ARE*

$$\tilde{A}'X + X\tilde{A} + X\tilde{B}_1\tilde{B}_1'X + \tilde{C}_1'\tilde{C}_1 = 0, \tag{3.33}$$

such that the matrix $\tilde{A} + \tilde{B}_1\tilde{B}_1'X$ is asymptotically stable. In particular, the matrix $\tilde{A}$ is asymptotically stable and the system $\tilde{\Sigma}_l^0$ has H_∞ norm strictly less than 1.

Conversely, assume that the matrix $\tilde{A} = A - B_1\bar{Q}^{-1}\bar{S}C_1$ is asymptotically stable and the system $\tilde{\Sigma}_l^0$ has H_∞ norm strictly less than 1. Then, there exists a solution $X \geq 0$ to the ARE in (3.19) such that the matrix $A^* = A + B_1\bar{Q}^{-1}[B_1'X - \bar{S}C_1]$ is asymptotically stable. In particular, the matrix A is asymptotically stable and the system Σ_l^0 (3.17) is strictly dissipative with respect to the supply rate $r_q(z,w)$ in (3.18).

Proof. Under the hypothesis, Theorem 3.8 implies there exists a solution $X \geq 0$ to the ARE (3.19) such that A^* is asymptotically stable. Note that the ARE in (3.19) is equivalent to the ARE in (3.33) and that $A^* = \tilde{A} + \tilde{B}_1\tilde{B}_1'X$. The equivalence of these AREs and matrices implies, by Theorem 3.8, that the related system $\tilde{\Sigma}_l^0$ has H_∞ norm strictly less than 1. To show that the matrix $\tilde{A}$ is asymptotically stable. Rewrite the ARE (3.33) as

$$\tilde{A}'X + X\tilde{A} + [X\tilde{B}_1 \quad \tilde{C}_1'][X\tilde{B}_1 \quad \tilde{C}_1']' = 0.$$

Since the matrix $A^* = \tilde{A} + [\tilde{B}_1 \quad 0][X\tilde{B}_1 \quad \tilde{C}_1']'$ is stable, the pair $(\tilde{A}, [X\tilde{B}_1 \quad \tilde{C}_1']')$ is detectable. Since $X \geq 0$ solves the above (linear) Lyapunov equation, we conclude that $\tilde{A}$ is an asymptotically stable matrix.

Conversely, under the hypothesis Theorem 3.8 implies there exists $X \geq 0$ solving the ARE (3.33) such that $\tilde{A} + \tilde{B}_1\tilde{B}_1'X$ is asymptotically stable. Since X also solves the ARE (3.19) such that A^* is asymptotically stable, Theorem 3.8 implies that the system Σ_l^0 is strictly dissipative with respect to the supply rate $r_q(z,w)$ in (3.18). To show that A is asymptotically stable, rewrite the ARE (3.19) as

$$A'X + XA + [[XB_1 - C_1'\bar{S}']\bar{Q}^{-1}W' \quad C_1'V'] \begin{bmatrix} W\bar{Q}^{-1}[B_1'X - \bar{S}C_1] \\ VC_1 \end{bmatrix},$$

where $W'W = \bar{Q} > 0,\ V'V = -R \geq 0$. Since

$$A^* = A + [B_1W^{-1} \quad 0] \begin{bmatrix} W\bar{Q}^{-1}[B_1'X - \bar{S}C_1] \\ VC_1 \end{bmatrix}$$

is asymptotically stable, the pair $(A, \begin{bmatrix} W\bar{Q}^{-1}[B_1'X - \bar{S}C_1] \\ VC_1 \end{bmatrix})$ is detectable. Since $X \geq 0$ solves the above Lyapunov equation, we conclude that A is asymptotically stable. $\square$

4 Dissipative Control Synthesis

We now turn to the dissipative control synthesis problem. We shall present the results for nonstrict and strict dissipativity cases for nonlinear (and linear) systems in terms of a PDI (ARI). We make the following definition regarding closed loop (zero state) detectability.

Definition 4.1 *We say that closed loop system* $\Sigma^{\mathbf{u}}$ *is (zero state)* detectable *if, under the control law* $\mathbf{u}$*, the conditions* $w = 0$ *and* $\lim_{t\to\infty} z(t) = 0$ *implies* $\lim_{t\to\infty} x(t) = 0$*.*

4.1 General Dissipative Control Synthesis

We consider the nonlinear systems in (2.1) with general supply rate function $r(z, w)$ in (2.2) satisfying condition (2.3).

Theorem 4.2 *Consider the system* Σ *in (2.1) and the supply rate* $r(z, w)$ *in (2.2) satisfying (2.3). Assume that there exists* $\mathbf{u}^{\mathrm{s}}(x) = K^s(x) \in \mathcal{S}$ *such that the closed loop system* $\Sigma^{\mathbf{u}^{\mathrm{s}}}$ *(2.1) is dissipative with respect to the supply rate* $r(z, w)$ *(2.2). Then there exists a solution* V *to the following PDI*

$$\begin{aligned} &\inf_{u\in\mathbf{R}^m} \sup_{w\in\mathbf{R}^d}\{\nabla_x V(x)\cdot(A(x) + B_1(x)w + B_2(x)u) \\ &\qquad\qquad -r(h_1(x, w, u), w)\} \le 0, \end{aligned} \tag{4.1}$$

where $h_1(x, w, u) = C_1(x) + D_{11}(x)w + D_{12}(x)u$*, such that* $V \ge 0$ *and* $V(0) = 0$*, in the viscosity sense.*

Conversely, assume that there exists $V \in C^1$ *solving the PDI (4.1) in the classical sense such that* $V \ge 0$ *and* $V(0) = 0$*. Suppose that the control law* $\mathbf{u}^* = \mathbf{u}^*(x)$ *attains minimum on the LHS of the PDI (4.1) i.e.*

$$\begin{aligned} &\mathbf{u}^* = \mathbf{u}^*(x) \\ &\in \arg\min_{u\in\mathbf{R}^m}\{\sup_{w\in\mathbf{R}^d}\{\nabla_x V(x)\cdot(A(x) + B_1(x)u + B_2(x)u) - r(h_1(x, w, u), w)\}\}. \end{aligned} \tag{4.2}$$

Then the closed loop $\Sigma^{\mathbf{u}^*}$ *is dissipative with respect to the supply rate* $r(z, w)$ *(2.2). Moreover if: (i) the supply rate* $r(z, w)$ *satisfies*

$$r(z, 0) \le -c_1|z|^2, \tag{4.3}$$

for all $z \ne 0$*, for some constant* $c_1 > 0$*, with* $r(0, 0) = 0$*, and (ii) the closed loop system* $\Sigma^{\mathbf{u}^*}$ *is zero state detectable, then* $\Sigma^{\mathbf{u}^*}$ *is asymptotically stable.*

Proof. Under the hypothesis, Theorem 3.2 implies there exists a solution $V \ge 0$, with $V(0) = 0$, to the PDI

$$\sup_{w\in\mathbf{R}^d}\{\nabla_x V\cdot(A(x) + B_1(x)w + B_2(x)K^s(x)) - r(h_1(x, w, K^s(x)), w)\} \le 0,$$

in the viscosity sense. Taking the minimum over $\mathbf{u}(x) = K(x) \in \mathcal{S}$ yields

$$\inf_{u\in\mathbf{R}^m} \sup_{w\in\mathbf{R}^d} \{\nabla_x V \cdot (A(x) + B_1(x)w + B_2(x)u) - r(h_1(x,w,u),w)\} \leq 0.$$

This proves the existence of a solution to the PDI (4.1).

Conversely, if a smooth solution $V \in C^1$ to the PDI (4.1) exists, such that $V \geq 0$, and $V(0) = 0$, then by using the control law $\mathbf{u}^*(x) = K^*(x)$ defined in (4.2), we get

$$\begin{aligned}
0 &\geq \int_0^T \sup_{w\in\mathbf{R}^d} \{\nabla_x V(x(t)) \cdot (A(x(t)) + B_1(x(t))w(t) + B_2(x(t))\mathbf{u}^*(x(t))) \\
&\quad - r(h_1(x(t), w(t), \mathbf{u}^*(x(t))), w(t))\} dt \\
&\geq \int_0^T (\nabla_x V \cdot (A(x(t)) + B_1(x(t))w(t) + B_2(x(t))\mathbf{u}^*(x(t))) \\
&\quad - r(h_1(x(t), w(t), \mathbf{u}^*(x(t))), w(t))) dt \\
&= V(x(T)) - V(x(0)) - \int_0^T r(h_1(x(t), w(t), \mathbf{u}^*(x(t))), w(t)) dt.
\end{aligned}$$

Since $V \geq 0$, the last inequality implies,

$$V(x(0)) \geq -\int_0^T r(h_1(x(t), w(t), \mathbf{u}^*(x(t))), w(t)) dt,$$

which shows that $\Sigma^{\mathbf{u}^*}$ is dissipative with respect to $r(z,w)$, with $\beta^{\mathbf{u}^*}(x) = V(x)$. Furthermore, by setting $w = 0$ in the last inequality we get

$$V(x(0)) \geq -\int_0^T r(h_1(x(t), 0, \mathbf{u}^*(x(t))), 0) dt.$$

By (4.3), the last inequality implies (see Remark 3.3) $h_1(x(t), 0, \mathbf{u}^*(t)) \to 0$ as $t \uparrow \infty$. By the zero state detectability assumption of $\Sigma^{\mathbf{u}^*}$ we conclude that $x(t) \to 0$ asymptotically. □

Remark 4.3 Suppose that the supply rate is quadratic as in (3.5) such that $\bar{Q}(x) > 0$, $\bar{R}(x) = \bar{S}(x)'\bar{Q}^{-1}(x)\bar{S}(x) - R > 0$ for all x, where $\bar{S}(x) = S + D_{11}(x)'R$. Then the PDI (4.1) can be written as (by direct evaluation)

$$\begin{aligned}
&\nabla_x V(x) \cdot (A(x) - \bar{B}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)C_1(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}C_1(x)) \\
&\quad - \tfrac{1}{2}\nabla_x V(x) \cdot (\bar{B}(x)\bar{E}^{-1}(x)\bar{B}(x)' - B_1(x)\bar{Q}^{-1}(x)B_1(x)') \cdot \nabla_x V(x)' \\
&\quad + \tfrac{1}{2}C_1(x)'\bar{D}(x)'\bar{R}(x)\bar{D}(x)C_1(x) \leq 0,
\end{aligned} \tag{4.4}$$

where $\bar{B}(x) = B_2(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)D_{12}(x)$, $\bar{E}(x) = D_{12}(x)'\bar{R}(x)D_{12}(x) > 0$ and $\bar{D}(x) = I - D_{12}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)$, and the optimal control in (4.2) is given by

$$\mathbf{u}^*(x) = -\bar{E}^{-1}(x)(\bar{B}(x)' \cdot \nabla_x V(x)' - D_{12}(x)'\bar{R}(x)C_1(x). \tag{4.5}$$

Remark 4.4 The condition (4.3) does not hold in the positive real (or passivity) case (here $r(z,0)=0$ for all w). Without the condition (4.3) holding, closed loop stability can be pursued in the following manner (see [16], [17]), which works in the quadratic supply rate case, including the positive real case. Assume that V is smooth and positive definite, i.e., $V(x)>0$ for all $x\neq 0$, with $V(0)=0$. The PDI (4.4) may be rewritten as

$$\nabla_x V(x)\cdot A^{\mathbf{u}^*}(x)+\tfrac{1}{2}[\nabla_x V(x)\cdot B_1(x)-C^{\mathbf{u}^*}(x)'\bar{S}(x)']\bar{Q}^{-1}(x)$$

$$[B_1(x)'\nabla_x V(x)'-\bar{S}(x)C^{\mathbf{u}^*}(x)]-C^{\mathbf{u}^*}(x)'RC^{\mathbf{u}^*}(x)=0,$$

where $A^{\mathbf{u}^*}(x)=A(x)+B_2(x)\mathbf{u}^*(x)$ and $C^{\mathbf{u}^*}(x)=C_1(x)+D_{12}\mathbf{u}^*(x)$. Then, along any trajectory $x(t)$ with $u(t)=\mathbf{u}^*(x(t))$ and $w=0$, we have (see also [16], [17])

$$\tfrac{dV}{dt}=-L(x(t))'L(x(t))+h_1'(x(t),0,\mathbf{u}^*(x(t)))Rh_1(x(t),0,\mathbf{u}^*(x(t))),$$

$$\leq h_1'(x(t),0,\mathbf{u}^*(x(t)))Rh_1(x(t),0,\mathbf{u}^*(x(t))),$$

for some bounded function $L(x)$. If $R\leq 0$, then the closed loop system is (Lyapunov) stable since V serves as a Lyapunov function. Thus, we conclude that a positive real system possesing a smooth and positive definite storage function is Lyapunov stable (see also [16], [8]). When $R<0$, the La Salle invariance principle implies that $x(\cdot)$ converges to the set $\{x : h_1(x,0,\mathbf{u}^*(x))=0\}$. Thus, if $\Sigma^{\mathbf{u}^*}$ is zero state detectable, we have $\lim_{t\to\infty}x(t)=0$, i.e. $\Sigma^{\mathbf{u}^*}$ is asymptotically stable.

Remark 4.5 If the system in (2.1) is linear, i.e. $A(x)=Ax, B_1(x)=B_1, B_2(x)=B_2, C_1(x)=C_1x, D_{11}(x)=D_{11}, D_{12}(x)=D_{12}$ where A, B_1, B_2, C_1, D_{11} and D_{12} are constant matrices, and the supply rate is quadratic as above, then the PDI (4.1) has an explicit expression given by $V(x)=\frac{1}{2}x'Xx$, where $X=X'\geq 0$ solves the following ARI

$$\begin{aligned}&X(A-\bar{B}\bar{E}^{-1}D_{12}'\bar{R}C_1-B_1\bar{Q}^{-1}\bar{S}C_1)+(A-\bar{B}\bar{E}^{-1}D_{12}'\bar{R}C_1-B_1\bar{Q}^{-1}\bar{S}C_1)'X\\&\quad -X(\bar{B}\bar{E}^{-1}\bar{B}'-B_1\bar{Q}^{-1}B_1')X+C_1'\bar{D}'\bar{R}\bar{D}C_1\leq 0,\end{aligned}\tag{4.6}$$

where $\bar{B}=B_2-B_1\bar{Q}^{-1}\bar{S}D_{12}$, $\bar{E}=D_{12}'\bar{R}D_{12}$ and $\bar{D}=I-D_{12}\bar{E}^{-1}D_{12}'\bar{R}$ and the optimal control law is linear in x given by $\mathbf{u}^*(x)=K^*x$, where

$$K^*=-\bar{E}^{-1}\bar{B}'X-D_{12}'\bar{R}C_1.\tag{4.7}$$

When $R<0$, closed loop asymptotic stability will be obtained if the matrices pair $(A+B_2K^*, C_1+D_{12}K^*)$ is completely observable. The ARI in (4.6) has infinitely many solutions and each of them will result in dissipative closed loop system. However, only one of them will result in a strict dissipativity property. This case will be discussed in more details in §4.3.

4.2 Nonlinear Strict Dissipation Synthesis

In this section we present control synthesis results for strict dissipativity, which are mainly based on the analysis results in §3.2. We consider the systems Σ (2.1) with quadratic supply rate $r_q(z,w)$ as in (3.5).

Theorem 4.6 *Consider the system in (2.1)and the quadratic supply rate $r_q(z,w)$ in (3.5). Assume that there exists a controller $\mathbf{u}^{\mathrm{s}} = K^s(x) \in \mathcal{S}$ such that the closed loop system $\Sigma^{\mathbf{u}^{\mathrm{s}}}$ (2.1) is strictly dissipative with respect to the supply rate $r_q(z,w)$ and the vector field $A^{\mathbf{u}^{\mathrm{s}}}(x) = A(x) + B_2(x)\mathbf{u}^{\mathrm{s}}(x)$ is strongly stable. Then the following statements hold true.*

(i) There exists a (locally bounded) lower semi continuous function $\hat{V} > 0$, with $\hat{V}(0) = 0$, satisfying the PDI

$$\begin{aligned}\inf_{u\in\mathbf{R}^m}\sup_{w\in\mathbf{R}^d}\{\nabla_x\hat{V}(x)\cdot(A(x)+B_1(x)w+B_2(x)u)\\ -r_q(h_1(x,w,u),w)+\delta\tfrac{1}{2}x'x\}\leq 0,\end{aligned}\tag{4.8}$$

for some $\delta > 0$, where $h_1(x,w,u) = C_1(x) + D_{11}(x)w + D_{12}(x)u$, in the viscosity sense.

(ii) There exists a (locally bounded) lower semi continuous function $V \geq 0$ with $V(0) = 0$, satisfying the PDE

$$\begin{aligned}\inf_{u\in\mathbf{R}^m}\sup_{w\in\mathbf{R}^d}\{\nabla_xV(x)\cdot(A(x)+B_1(x)w+B_2(x)u)\\ -r_q(h_1(x,u,w),w)\}=0,\end{aligned}\tag{4.9}$$

in the viscosity sense.

(iii) Assume that $D_{12}(x) = I$. If the solution V to (4.9) is smooth (i.e. C^1) we have the following stability results.

a. Assume that $-A^{\times\times}(x) \triangleq -(A(x)-B_2(x)C_1(x))$ is Lyapunov stable. If $V(x) > 0$ for all $x \neq 0$, with $V(0) = 0$, and V is a Lyapunov function for $-A^{\times\times}(x)$, then the vector field

$$\begin{aligned}A^*(x)\triangleq A(x)-B_2(x)C_1(x)-([B_2(x)-B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)]\bar{R}^{-1}(x)\\ [B_2(x)-B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)]'-B_1(x)\bar{Q}^{-1}(x)B_1(x)')\cdot\nabla_xV(x)'\end{aligned}\tag{4.10}$$

is stable in the sense that $V(x)$ is a Lyapunov function for $A^*(x)$. In particular, if $-A^{\times\times}(x)$ is exponentially stable, then so is $A^*(x)$. Thus, V is a stabilizing solution to the PDI (4.8).

b. Assume that $A^{\times\times}(x) \triangleq (A(x) - B_2(x)C_1(x))$ *is asymptotically stable. Then* $V = 0$ *is a stabilizing solution to the PDI (4.8).*

PROOF. Under the hypothesis, Theorem 3.6 implies there exists a solution $\hat{V} > 0$, with $\hat{V}(0) = 0$, to the PDI

$$\sup_{w \in \mathbf{R}^d} \{\nabla_x \hat{V}(x) \cdot (A(x) + B_1(x)w + B_2(x)K^s(x)) - r_q(h_1(x, w, K^s(x)), w) + \delta\frac{1}{2}x'x\} \leq 0,$$

in the viscosity sense. Taking the infimum over $\mathbf{u} = K(x) \in \mathcal{S}$ yields

$$\inf_{u\mathbf{R}^m} \sup_{w \in \mathbf{R}^d} \{\nabla_x \tilde{V}(x) \cdot (A(x) + B_1(x)w + B_2(x)u) - r_q(h_1(x, w, u), w) + \delta\frac{1}{2}x'x\} \leq 0.$$

This proves the existence of a solution to the PDI (4.8). Next, define the function V by

$$V(x) = \sup_{T \geq 0, w \in L_2([0,T], \mathbf{R}^d)}\{-\int_0^T r_q(h_1(x(t), w(t), K^s(x(t))), w(t))dt : x(0) = x\}.$$

Clearly we have $0 \leq V \leq \beta^{\mathbf{u}^{\mathbf{s}}}$. By Theorem 3.2, V solves the PDI

$$\sup_{w \in \mathbf{R}^d} \{\nabla_x V(x) \cdot (A(x) + B_1(x)w + B_2(x)K^s(x)) - r_q(h_1(x, w, K^s(x)), w)\} \leq 0,$$

in the viscosity sense. Now, taking the minimum over $\mathbf{u} = K(x) \in \mathcal{S}$ yields

$$\inf_{u\mathbf{R}^m} \sup_{w \in \mathbf{R}^d} \{\nabla_x V(x) \cdot (A(x) + B_1(x)w + B_2(x)u) - r(h_1(x, w, u), w)\} \leq 0.$$

This proves that V is a viscosity supersolution of the PDE (4.9). Proof that V is also a viscosity subsolution is given in the Appendix (see also [5], [25]). This completes the proof of assertion (ii). Next, if V is smooth, evaluation of the inf and sup operations in the PDE (4.9), yields, with $D_{12}(x) = I$,

$$\nabla_x V(x) \cdot (A(x) - B_2(x)C_1(x)) - \tfrac{1}{2}\nabla_x V(x) \cdot ([B_2(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)]\bar{R}^{-1}(x)$$

$$([B_2(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)]' - B_1(x)\bar{Q}^{-1}(x)B_1(x)') \cdot \nabla_x V(x)' = 0.$$

This PDI can be reexpressed in terms of the vector fields $A^*(x)$ and $A^{\times\times}(x)$ as follows

$$\nabla_x V(x) \cdot A^*(x) = \nabla_x V(x) \cdot (-A^{\times\times}(x)).$$

If $V > 0$ is a Lyapunov function for $-A^{\times\times}(x)$ (i.e. $\nabla_x V(x) \cdot (-A^{\times\times}(x)) \leq 0$ for all x), then, along any trajectory produced by $A^*(x)$ we have

$$\begin{aligned} \tfrac{dV}{dt}(x(t)) &= \nabla_x V(x(t)) \cdot A^*(x(t)) \\ &= \nabla_x V(x(t)) \cdot (-A^{\times\times}(x(t))) \\ &\leq 0. \end{aligned}$$

Assertion concerning exponential stability can be verified in the same manner. This proves assertion (iiia).

If $A^{\times\times}(x) \triangleq (A(x) - B_2(x)C_1(x))$ is asymptotically stable, then with $V = 0$ (which solves the PDE (4.9)) we have

$$A^*(x) = A^{\times\times}(x),$$

and hence, $A^*(x)$ is an asymptotically stable vector field. This completes the proof. $\square$

Conversely, we have the following results.

Theorem 4.7 *Assume that there exists $V \in C^1$ solving the PDE (4.9) in the classical sense with $V \geq 0$ and $V(0) = 0$, such that the vector field*

$$\begin{aligned} A^*(x) = {} & A(x) - \bar{B}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)C_1(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}C_1(x) \\ & -\tfrac{1}{2}\nabla_x V(x) \cdot (\bar{B}(x)\bar{E}^{-1}(x)\bar{B}(x)' - B_1(x)\bar{Q}^{-1}(x)B_1(x)'), \end{aligned} \tag{4.11}$$

where $\bar{B}(x) = B_2(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)D_{12}(x)$, $\bar{E}(x) = D_{12}(x)'\bar{R}(x)D_{12}(x)$ and $\bar{D}(x) = I - D_{12}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)$, is strongly stable. By employing the control law $\mathbf{u}^*(x)$ defined by

$$\mathbf{u}^*(x) = -\bar{E}^{-1}(x)(\bar{B}(x)' \cdot \nabla_x V(x)' - D_{12}(x)'\bar{R}(x)C_1(x). \tag{4.12}$$

(which attains the minimum on the LHS of the PDE (4.9)) the closed loop vector field $A^{\mathbf{u}^*}(x) = A(x) + B_2(x)\mathbf{u}^*(x)$ is weakly stable and the closed loop system $\Sigma^{\mathbf{u}^*}$ (2.1) is dissipative with respect to $r_q(z,w)$. Moreover, if the vector field $w^{u^*}(x) = \bar{Q}^{-1}(x)[B_1(x)'\nabla_x V(x)' - \bar{S}(x)C^{\mathbf{u}^*}(x)]$, with $C^{\mathbf{u}^*}(x) = C_1(x) + D_{12}(x)\mathbf{u}^*(x)$, satisfies $|w^{u^*}(x)| \leq \alpha|x|$, for all x, for some $\alpha > 0$, then the dissipativity is strict.

Proof. The results follow by evaluating the PDE (4.9) to get the PDI as in (4.4), with equality, and rewriting the resulting PDE as follows

$$\begin{aligned} & \nabla_x V(x) \cdot A^{\mathbf{u}^*}(x) + \tfrac{1}{2}[\nabla_x V(x) \cdot B_1(x) - C^{\mathbf{u}^*}(x)'\bar{S}(x)']\bar{Q}^{-1}(x) \\ & \qquad [B_1(x)'\nabla_x V(x)' - \bar{S}(x)C^{\mathbf{u}^*}(x)] - C^{\mathbf{u}^*}(x)'RC^{\mathbf{u}^*}(x) = 0, \end{aligned}$$

with $A^{\mathbf{u}^*}(x) = A(x) + B_2(x)\mathbf{u}^*(x)$, $C^{\mathbf{u}^*}(x) = C_1(x) + D_{12}(x)\mathbf{u}^*(x)$, and then by performing calculations similar to those in the proof of the converse part of Theorem 3.6. $\square$

4.3 Linear Strict Dissipation Synthesis

We present results analoguous to those in the previous section for linear systems with quadratic supply rate. In particular, we characterize the strict dissipativity property in terms of the stabilizing solution to an ARE. These results extend known ones in the literature on H_∞ control or positive real control (see [13], [23], [26]).

We consider linear models given by

$$\Sigma_l : \begin{cases} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \quad x(0) = x_0, \\ \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t), \end{cases} \tag{4.13}$$

in which $A \in \mathbf{R}^{n\times n}, B_1 \in \mathbf{R}^{n\times d}, B_2 \in \mathbf{R}^{n\times m}, C_1 \in \mathbf{R}^{q\times n}, D_{11} \in \mathbf{R}^{q\times d}, D_{12} \in \mathbf{R}^{q\times m}$ are constant matrices, with D_{12} satisfying $D_{12}' D_{12} = E_1 > 0$.

We consider a quadratic supply rate $r_q(z, w)$ as in (3.18) satisfying $R \leq 0$, $\bar{Q} = Q + SD_{11} + D_{11}'S' + D_{11}'RD_{11} > 0$ and $\bar{R} = \bar{S}'\bar{Q}^{-1}\bar{S} - R > 0$, where $\bar{S} = S + D_{11}'R$. We assume that $\text{rank}\{M(j\omega)\} = n + m$, where

$$M(j\omega) = \begin{bmatrix} A - j\omega I & B_2 \\ \\ C_1 & D_{12} \end{bmatrix}. \tag{4.14}$$

Theorem 4.8 *Assume that there exists a linear feedback policy $\mathbf{u}^{\mathbf{s}}(x) = K^s x$, such that $\Sigma_l^{\mathbf{u}^{\mathbf{s}}}$ is strictly dissipative with respect to the quadratic supply rate $r_q(z, w)$ (3.18). Then, there exists a solution $X \geq 0$ to the following ARE*

$$\begin{aligned} &X(A - \bar{B}\bar{E}^{-1}D_{12}'\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1) + (A - \bar{B}\bar{E}^{-1}D_{12}'\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1)'X \\ &\quad -X(\bar{B}\bar{E}^{-1}\bar{B}' - B_1\bar{Q}^{-1}B_1')X + C_1'\bar{D}'\bar{R}\bar{D}C_1 = 0, \end{aligned} \tag{4.15}$$

such that the matrix

$$A^* = (A - \bar{B}\bar{E}^{-1}D_{12}'\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1) - (\bar{B}\bar{E}^{-1}\bar{B}' - B_1\bar{Q}^{-1}B_1')X \tag{4.16}$$

is asymptotically stable.

Conversely, suppose there exists a solution X to the ARE (4.15) such that $X \geq 0$ and the matrix A^* in (4.16) is asymptotically stable. Then, under the control law $\mathbf{u}^*(x) = K^* x$, where

$$K^* = -\bar{E}^{-1}(B_2 - B_1\bar{Q}^{-1}\bar{S}D_{12})'X - D_{12}\bar{R}C_1, \tag{4.17}$$

the closed loop matrix $A^{u^*} = A + B_2 K^*$ is asymptotically stable, and the closed loop system $\Sigma_l^{\mathbf{u}^*}$ in (4.13) is strictly dissipative with respect to the supply rate $r_q(z, w)$ (3.18).

Proof. (Necessity) We follow the proofs in [23] and in [26]. Under the hypothesis, the closed loop system

$$\begin{aligned} \dot{x}(t) &= A^{u^s} x(t) + B_1 w(t), \quad x(0) = x_0, \\ \\ z(t) &= C^{u^s} x(t) + D_{11} w(t), \end{aligned}$$

where $A^{u^s} = A+B_2K^s$, $C^{u^s} = C_1+D_{12}K^s$ is strictly dissipative with respect to the supply rate (3.18), such that the matrix A^{u^s} is asymptotically stable. Corollary 3.9 implies that the related system

$$\begin{aligned} \dot{x}(t) &= \tilde{A}^s x(t) + \tilde{B}_1 w(t),\ x(0) = x_0, \\ z(t) &= \tilde{C}_1^s x(t), \end{aligned}$$

where $\tilde{A}^s = A^{u^s} - B_1\bar{Q}^{-1}\bar{S}C^{u^s}$, $\tilde{B}_1 = B_1\bar{Q}^{-\frac{1}{2}}$, and $\tilde{C}_1^s = (\bar{S}'\bar{Q}^{-1}\bar{S} - R)^{\frac{1}{2}}C^{u^s}$ has H_∞ norm strictly less than 1. Now, Theorem 2.1 in [23] implies there exists a matrix $\tilde{X} > 0$ solving the following ARI

$$(\tilde{A}^s)'\tilde{X} + \tilde{X}\tilde{A}^s + \tilde{X}\tilde{B}_1\tilde{B}_1'\tilde{X} + (\tilde{C}_1^s)'\tilde{C}_1^s < 0.$$

This ARI can be rewritten in terms of the parameters of the original system as follows

$$(A^{u^s})'\tilde{X} + \tilde{X}A^{u^s} + [\tilde{X}B_1 - (C^{u^s})'\bar{S}']\bar{Q}^{-1}[B_1'\tilde{X} - \bar{S}C^{u^s}] - (C^{u^s})'RC^{u^s} < 0.$$

Write $W = \tilde{X}^{-1} > 0$. By multiplying both sides of the above inequality from left and right by W, and rearranging the resulting inequality we get

$$\begin{aligned} &W(A - \bar{B}\bar{E}^{-1}D_{12}'\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1)' + (A - \bar{B}\bar{E}^{-1}D_{12}'\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1)W \\ &\quad -(\bar{B}\bar{E}^{-1}\bar{B}' - B_1\bar{Q}^{-1}B_1') + WC_1'\bar{D}'\bar{R}\bar{D}C_1W + Z\bar{E}^{-1}Z' < 0, \end{aligned}$$

where $\bar{B} = B_2 - B_1\bar{Q}^{-1}\bar{S}D_{12}$, $\bar{E} = D_{12}'\bar{R}D_{12} > 0$, $\bar{D} = I - D_{12}\bar{E}^{-1}D_{12}'\bar{R}$ and $Z = W(K^{s\prime} + C_1'\bar{R}D_{12}\bar{E}^{-1})\bar{E} + \bar{B}$. Since $\bar{E} > 0$, the above inequality implies

$$\begin{aligned} &W(A - \bar{B}\bar{E}^{-1}D_{12}'\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1)' + (A - \bar{B}\bar{E}^{-1}D_{12}'\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1)W \\ &\quad -(\bar{B}\bar{E}^{-1}\bar{B}' - B_1\bar{Q}^{-1}B_1') + WC_1'\bar{D}'\bar{R}\bar{D}C_1W < 0. \end{aligned} \tag{4.18}$$

Under the assumption rank$\{M(j\omega)\} = n+m$, where $M(j\omega)$ is given in (4.14), the relation

$$\begin{bmatrix} \hat{A} - j\omega I & \bar{B}\bar{E}^{-\frac{1}{2}} \\ \hat{C} & D_{12}\bar{E}^{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} I & -B_1\bar{Q}^{-1}\bar{S} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - j\omega & B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\bar{E}^{-1}D_{12}'\bar{R}C_1 & \bar{E}^{-\frac{1}{2}} \end{bmatrix},$$

where $\hat{A} = A - \bar{B}\bar{E}^{-1}D_{12}'\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1$, $\hat{C} = \bar{D}C_1$ implies that the pair $(\hat{A}, \hat{C})$ has no unobservable modes on the imaginary axis. This allows us to perform the following decomposition (using appropriate coordinate transformation)

$$\hat{A} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad \hat{C} = [C_{11} \quad 0],$$

where A_{22} is stable and the pair $(-A_{11}, C_{11})$ is detectable (since A_{11} contains either modes that are observable through C_{11} or modes that are anti stable). In the new coordinates, write $\bar{B} = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}$, $B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}$ and $W = \begin{bmatrix} W_{11} & W_{12}' \\ W_{12} & W_{22} \end{bmatrix}$. Substituting these matrices into the ARI (4.18), and taking the (1,1) block inequality yields

$$W_{11}A_{11}' + A_{11}W_{11} - B_{21}\bar{E}^{-1}B_{21}' + B_{11}\bar{Q}^{-1}B_{11}' + W_{11}C_{11}'\bar{R}C_{11}W_{11} < 0,$$

that is there exists a positive definite matrix P such that

$$W_{11}A_{11}' + A_{11}W_{11} - B_{21}\bar{E}^{-1}B_{21}' + B_{11}\bar{Q}^{-1}B_{11}' + W_{11}C_{11}'\bar{R}C_{11}W_{11} + P = 0. \tag{4.19}$$

Since $(-A_{11}, C_{11})$ is detectable, the comparison theorem of Riccati equations (Theorem 2.2 of [24]) implies there exists a matrix $W_{11E} \geq W_{11} > 0$ solving the ARE

$$W_{11E}A_{11}' + A_{11}W_{11E} - B_{21}\bar{E}^{-1}B_{21}' + B_{11}\bar{Q}^{-1}B_{11}' + W_{11E}C_{11}'\bar{R}C_{11}W_{11E} = 0, \tag{4.20}$$

with $W_{11E} \geq W_{11}$ such that all eigen values of $-(A_{11} + W_{11E}C_{11}'\bar{R}C_{11})$ are in the closed LHP of the complex plane. We shall show that $-(A_{11} + W_{11E}C_{11}'\bar{R}C_{11})$ is, in fact, asymptotically stable. Substracting (4.20) from (4.19) yields

$$Z(-(A_{11} + W_{11E}C_{11}'\bar{R}C_{11}))' + (-(A_{11} + W_{11E}C_{11}'\bar{R}C_{11}))Z + P + ZC_{11}'\bar{R}C_{11}Z = 0,$$

with $Z = W_{11E} - W_{11} \geq 0$, $P + ZC_{11}'\bar{R}C_{11}Z > 0$. By applying the standard (linear) Lyapunov theorem, we conclude that $(-(A_{11} + W_{11E}C_{11}'\bar{R}C_{11}))$ is asymptotically stable and $W_{11E} > W_{11}$. Now, straightforward calculation shows that the matrix $X_{11E} \geq 0$ defined by

$$X_E = \begin{bmatrix} W_{11E}^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

satisfies

$$\begin{aligned} & X_E(A - \bar{B}\bar{E}^{-1}D_{12}'\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1) + (A - \bar{B}\bar{E}^{-1}D_{12}'\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1)'X_E \\ & -X_E(\bar{B}\bar{E}^{-1}\bar{B}' - B_1\bar{Q}^{-1}B_1')X_E + C_1'\bar{D}'\bar{R}\bar{D}C_1 \\ & \qquad = A_{11}'W_{11E}^{-1} + W_{11E}^{-1}A_{11} - W_{11E}^{-1}(B_{21}\bar{R}^{-1}B_{21}' - B_{11}\bar{Q}^{-1}B_{11}')W_{11E}^{-1} + C_{11}'\bar{R}C_{11} \\ & \qquad = 0. \end{aligned}$$

Moreover, we have

$$\begin{aligned} A^* &= (A - \bar{B}\bar{E}^{-1}D_{12}'\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1) - (\bar{B}\bar{E}^{-1}\bar{B}' - B_1\bar{Q}^{-1}B_1')X_E \\ &= \begin{bmatrix} A_{11} - (B_{21}\bar{E}^{-1}B_{21}' - B_{11}\bar{Q}^{-1}B_{11}')W_{11E}^{-1} & 0 \\ A_{21} - (B_{22}\bar{E}^{-1}B_{21}' - B_{12}\bar{Q}^{-1}B_{11}')W_{11E}^{-1} & A_{22} \end{bmatrix}. \end{aligned}$$

Since A_{22} is asymptotically stable and the matrix $A_{11} - (B_{21}\bar{E}^{-1}B_{21}' - B_{11}\bar{Q}^{-1}B_{11}')W_{11E}^{-1}$ $= W_{11E}(-(A_{11} + W_{11E}C_{11}'\bar{R}C_{11}))W_{11E}^{-1}$ is also asymptotically stable, the above expression shows that the matrix A^* is asymptotically stable. This proves that X_E is the required stabilizing solution to the ARE (4.15).

(Sufficiency) Write $A^{u^*} = A + B_2K^*$, $C^{u^*} = C_1 + D_{12}K^*$, where

$$K^* = -\bar{E}^{-1}[B_2 - B_1\bar{Q}^{-1}\bar{S}D_{12}]'X - D_{12}'\bar{R}C_1.$$

Then the ARE in (4.15) can be rewritten as

$$A^{u^*\prime}X + XA^{u^*} + [XB_1 - C^{u^*\prime}\bar{S}']\bar{Q}^{-1}[B_1'X - \bar{S}C^{u^*}] - C^{u^*\prime}RC^{u^*} = 0.$$

Moreover, the matrix $A^{u^*} + B_1\bar{Q}^{-1}[B_1'X - \bar{S}C^{u^*}] = A^*$ is asymptotically stable. Therefore, by Theorem 3.8 the closed loop system $\Sigma_l^{\mathbf{u}^*}$ is strictly dissipative with respect to $r_q(z, w)$ in (3.18) and the closed loop matrix A^{u^*} is asymptotically stable. $\square$

Remark 4.9 When the expression on the LHS of the PDI (4.1) is not strictly concave in w, then the extremum w^* , when they exist, may not be unique. The PDI may then break down into a set of equations. We illustrate this situation for linear systems case. Suppose that A, B_1, B_2, C_1 are constant matrices, and $D_{11} = 0$. The supply rate is quadratic as in (3.18). Assume that $S = I$, and $\bar{Q} = Q = 0$. Then the PDI (4.1) becomes

$$\inf_{u\in\mathbf{R}^m}\sup_{w\in\mathbf{R}^d}\{\nabla_x V(x)\cdot(A + B_1w + B_2u) - \tfrac{1}{2}(2w'C_1x + x'C_1'RC_1x)\} \leq 0,$$

which is linear in w. Now assuming that $V(x) = \frac{1}{2}Xx^2$, and $u = Kx$, for some constant matrices $X \geq 0$, K, this PDI breaks down into two equations

$$\begin{aligned} B_1X &= (C_1 + K), \quad X \geq 0, \\ \min_K\{X(A + B_2K) &+ (A + B_2K)'X - (C_1 + K)'R(C_1 + K)\} \leq 0. \end{aligned}$$

As an example, suppose that A, B_1, B_2, C_1 are scalars, and assume that $A - B_2C_1 < 0$, $B_1(B_2 + 2B_1) > 0$. Then we have, after subtituting $K = B_1X - C_1$ into the last inequality, the following optimization problem

$$\min_X\{2(A - B_2C_1)X + B_1(B_2 + 2B_1)X^2\} \leq 0, \quad X \geq 0,$$

which has the unique minimizer given by $X^* = -(A - B_2C_1)/(B_1(B_2 + 2B_1)) > 0$. The optimal feedback policy is then given by $\mathbf{u}^* = K^*x$, where $K^* = B_1X^* - C_1$.

5 Local Solutions

In the H_∞ control case, it is known that the control problem for the nonlinear systems (2.1) is locally solvable if the problem for its linearized model is solvable [27], [28], [29]. In this section we present an extension of this result to the more general quadratic supply rate case. We consider the nonlinear systems Σ in (2.1) and its linearization given by Σ_{lin}

$$\Sigma_{lin}: \begin{cases} \dot{x}(t) &= A_l x(t) + B_{1l} w(t) + B_{2l} u(t), \quad x(0) = x_0, \\ \\ z(t) &= C_{1l} x(t) + D_{11l} w(t) + D_{12l} u(t), \end{cases} \tag{5.1}$$

where $A_l = \frac{\partial A}{\partial x}(0)$, $C_{1l} = \frac{\partial C_1}{\partial x}(0)$, $B_{1l} = B_1(0)$, $B_{2l} = B_2(0)$, $D_{11l} = D_{11}(0)$ and $D_{12l} = D_{12}(0)$. We denote $D'_{12l} D_{12l} = E_{1l} > 0$. As in the previous section, we consider a quadratic supply rate

$$r_q(z,w) = \frac{1}{2}(w'Qw + 2w'Sz + z'Rz), \tag{5.2}$$

where $Q,\ S,\ R$ are constant matrices, with $Q,\ R$ symmetric, satisfying $R \leq 0$. We assume that

$$\bar{Q}_l = Q + SD_{11l} + D'_{11l}S' + D'_{11l}RD_{11l} > 0, \quad \bar{R}_l = \bar{S}'_l \bar{Q}_l^{-1} \bar{S}_l - R > 0,$$

where $\bar{S}_l = S + D'_{11l}R$, and that

$$\bar{Q}(x) = Q + SD_{11}(x) + D_{11}(x)'S' + D_{11}(x)'RD_{11}(x) > 0,$$

$$\bar{R}(x) = \bar{S}(x)'\bar{Q}^{-1}(x)\bar{S}(x) - R > 0,$$

for all x, where $\bar{S}(x) = S + D_{11}(x)'R$.

We consider the manifold $T^*\mathbf{R}^n$ (dimension$(T^*\mathbf{R}^n) = 2n$), with coordinates $x_1, \ \ldots\ ,$ $x_n,\ p_1,\ \ldots\ , p_n$, endowed with a natural symplectic form $\bar{\omega}$. In the next theorem we shall need the following Hamiltonian function $H(x,p)$ defined on $T^*\mathbf{R}^n$

$$\begin{aligned} H(x,p) &= p' \cdot (A(x) - \bar{B}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)C_1(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}C_1(x)) \\ &\quad -\tfrac{1}{2}p \cdot (\bar{B}(x)\bar{E}^{-1}(x)\bar{B}(x)' - B_1(x)\bar{Q}^{-1}(x)B_1(x)') \cdot p \\ &\quad +\tfrac{1}{2}C_1(x)'\bar{D}(x)'\bar{R}(x)\bar{D}(x)C_1(x) \end{aligned} \tag{5.3}$$

where $\bar{B}(x) = B_2(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)D_{12}(x)$, $\bar{E}(x) = D_{12}(x)'\bar{R}(x)D_{12}(x)$ and $\bar{D}(x) = I - D_{12}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)$.

Theorem 5.1 *Consider the nonlinear system Σ (2.1) and its linearization Σ_{lin} (5.1) and assume that*

$$rank\left(\begin{bmatrix} A_l - j\omega I & B_{2l} \\ \\ C_{1l} & D_{12l} \end{bmatrix}\right) = n + m, \tag{5.4}$$

for all real ω. Suppose there exists a linear feedback control law $\mathbf{u^s}(x) = K^s x$, such that the linearized system $\Sigma_{lin}^{\mathbf{u^s}}$ (5.1) is strictly dissipative with respect to the quadratic supply *rate $r_q(z,w)$ in (5.2), and the closed loop matrix $A_l^{u^s} = A_l + B_{2l}K^s$ is asymptotically stable. Then there exists a smooth solution V to the PDE*

$$
\begin{aligned}
&\nabla_x V(x)\cdot(A(x) - \bar{B}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)C_1(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}C_1(x)) \\
&\quad -\tfrac{1}{2}\nabla_x V(x)\cdot(\bar{B}(x)\bar{E}^{-1}(x)\bar{B}(x)' - B_1(x)\bar{Q}^{-1}(x)B_1(x)')\cdot\nabla_x V(x)' \\
&\quad +\tfrac{1}{2}C_1(x)'\bar{D}(x)'\bar{R}(x)\bar{D}(x)C_1(x) = 0
\end{aligned} \tag{5.5}
$$

with $V(x) \geq 0$, $V(0) = 0$, on a neighbourhood W of $x_0 = 0$, such that the vector fields

$$
\begin{aligned}
A^*(x) &= A(x) - \bar{B}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)C_1(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}C_1(x) \\
&\quad -\tfrac{1}{2}(\bar{B}(x)\bar{E}^{-1}(x)\bar{B}(x)' - B_1(x)\bar{Q}^{-1}(x)B_1(x)')\cdot\nabla_x V(x)'
\end{aligned} \tag{5.6}
$$

and

$$A^{\mathbf{u}^*}(x) = A(x) + B_2(x)\mathbf{u}^*(x), \tag{5.7}$$

where

$$\mathbf{u}^*(x) = -\bar{E}^{-1}(x)(B_2(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)D_{12}(x))'\nabla_x V(x)' - D_{12}(x)\bar{R}(x)C_1(x), \tag{5.8}$$

are locally exponentially stable. Furthermore, the closed loop systems $\Sigma^{\mathbf{u}^*}$ (2.1) is locally dissipative with respect to the supply rate $r_q(z,w)$ in (5.2) (i.e., the inequality (2.4) holds for all $T \geq 0$, for all $w \in L_2([0,T],\mathbf{R}^d)$ such that the corresponding trajectory $x(\cdot)$ does not leave W).

Proof. Under the hypothesis, Theorem 4.8 implies there exists a nonnegative matrix X solving the ARE

$$
\begin{aligned}
&X(A_l - \bar{B}_l\bar{E}_l^{-1}D_{12l}'\bar{R}_lC_{1l} - B_{1l}\bar{Q}_l^{-1}\bar{S}_lC_{1l}) + (A_l - \bar{B}_l\bar{E}_l^{-1}D_{12l}'\bar{R}_lC_{1l} - B_{1l}\bar{Q}_l^{-1}\bar{S}_lC_{1l})'X \\
&\quad -X(\bar{B}_l\bar{E}_l^{-1}\bar{B}_l' - B_{1l}\bar{Q}_l^{-1}B_{1l}')X + C_{1l}'\bar{D}_l'\bar{R}_l\bar{D}_lC_{1l} = 0.
\end{aligned} \tag{5.9}
$$

such that the matrix

$$A_l^* = (A_l - \bar{B}_l\bar{E}_l^{-1}D_{12l}'\bar{R}_lC_{1l} - B_{1l}\bar{Q}_l^{-1}\bar{S}_lC_{1l}) - X(\bar{B}_l\bar{E}_l^{-1}\bar{B}_l' - B_{1l}\bar{Q}_l^{-1}B_{1l}') \tag{5.10}$$

and

$$A_l^{u^*} = (A_l - B_{2l}D_{12l}\bar{R}_lC_{1l}) - B_{2l}\bar{E}_l^{-1}[B_{2l} - B_{1l}\bar{Q}_l^{-1}\bar{S}_lD_{12l}]' \tag{5.11}$$

are asymptotically stable, in which expressions $\bar{B}_l = B_{2l} - B_{1l}\bar{Q}_l^{-1}\bar{S}_lD_{12l}$, $\bar{E}_l = D_{12l}'\bar{R}_lD_{12l}$ and $\bar{D}_l = I - D_{12l}\bar{E}_l^{-1}D_{12l}'\bar{R}_l$. Define the Hamiltonian matrix

$$
Ham = \begin{bmatrix} (A_l - \bar{B}_l\bar{E}_l^{-1}D_{12l}'\bar{R}_lC_{1l} - B_{1l}\bar{Q}_l^{-1}\bar{S}_lC_{1l}) & \bar{B}_l\bar{E}_l^{-1}\bar{B}_l' \\ C_{1l}'\bar{D}_l'\bar{R}_l\bar{D}_lC_{1l} & -(A_l - \bar{B}_l\bar{E}_l^{-1}D_{12l}'\bar{R}_lC_{1l} - B_{1l}\bar{Q}_l^{-1}\bar{S}_lC_{1l})' \end{bmatrix}.
$$

Direct calculation yields

$$\begin{bmatrix} I & 0 \\ X & I \end{bmatrix} Ham \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$$

$$= \begin{bmatrix} A_l^* & \bar{B}_l \bar{E}_l^{-1} \bar{B}_l' \\ 0 & -(A_l^*)' \end{bmatrix}.$$

Therefore, the eigenvalues of Ham are symmetric with respect to the imaginary axis. Since the matrix A_l^*, with dimension n, is asymptotically stable, we conclude Ham has no purely imaginary eigenvalues. Moreover, the relation

$$Ham \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} A_l^*$$

implies that

$$\mathrm{span}(\begin{bmatrix} I \\ X \end{bmatrix}) = \text{stable eigenspace of } Ham. \tag{5.12}$$

Next, consider the Hamiltonian vector field X_H on $T^*\mathbf{R}^n$, associated with the Hamiltonian function $H(x,p)$ in (5.3), defined by

$$X_H(x,p) = \begin{bmatrix} \frac{\partial H'}{\partial p} \\ \\ -\frac{\partial H'}{\partial x} \end{bmatrix} (x,p).$$

Note that $(0,0)$ is an equilibrium point of X_H. By direct calculation, we get the linearization matrix of X_H arround $(0,0)$ given by

$$DX_H(0,0) = \begin{bmatrix} \frac{\partial^2 H}{\partial x \partial p} & \frac{\partial^2 H}{\partial p^2} \\ \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial p \partial x} \end{bmatrix}$$

$$= Ham.$$

Thus, the linearization of X_H arround $(0,0)$ equals Ham. Since Ham has no pure imaginary eigenvalues, X_H is hyperbolic in $(0,0)$. Proposition A.4 of [28] says there exists a stable (invariant) manifold N^- of X_H, with dimension$(N^-) = n$, passing through $(0,0)$, such that the restriction of X_H on N^- is asymptotically stable (with regard to $(0,0)$). Moreover, N^- is tangent at $(0,0)$ to the stable eigen space of $DX_H(0,0)$, i.e. (using (5.12))

$$\begin{aligned} T_{(0,0)}N^- &= \text{stable eigenspace of } Ham \\ &= \mathrm{span}(\begin{bmatrix} I \\ X \end{bmatrix}) \end{aligned} \tag{5.13}$$

By Proposition A.5 of [28], N^- is also a Lagrangian submanifold of $T^*\mathbf{R}^n$ (i.e., N^- has dimension equals n and the restriction of the symplectic form $\bar{\omega}$ on N^- equals 0). From (5.13), we see that N^- may be parametrized by the x coordinate locally arround $(0,0)$. By Proposition A.6 of [28], there exists a smooth function V, with $V(0)=0$, $\nabla_x V(0)=0$, defined on a neighborhood W^0 of $x=0$ such that for all $x\in W^0$, N^- is given by

$$N^- = \{(x,p) : p = \frac{\partial V}{\partial x}, \quad x \in W^0\}.$$

Now we shall show that V solves the PDE (5.5) locally. Note that since the Lie derivative $\mathcal{L}_{X_H} H = \frac{\partial H}{\partial x}\cdot\frac{\partial H}{\partial p}' + \frac{\partial H}{\partial p}\cdot(-\frac{\partial H}{\partial x}') = 0$, $H(x,p)$ is constant along any integral trajectory of X_H. Since the restriction of X_H on N^- is asymptotically stable, X_H produces a trajectory that goes to $(0,0)$ as $t\to\infty$ for any initial condition (x_0,p_0) in N^-. In particular, along any trajectory which starts from $(x_0, \nabla_x V(x_0))$, $x_0 \in W^0$, we have

$$\begin{aligned} H(x_0, \nabla_x V(x_0)) &= H(x(t), p(t)) \\ &= \lim\nolimits_{t\to\infty} H(x(t), p(t)) \\ &= H(\lim\nolimits_{t\to\infty} x(t), \lim\nolimits_{t\to\infty} p(t)) \ \text{(by continuity of } H) \\ &= H(0,0) \\ &= 0. \end{aligned} \tag{5.14}$$

Thus $H(x, \nabla_x V(x)) = 0$ on W^0. This shows that V solves the PDE (5.5) on W^0. Next, note that the equation (5.14) together with (5.13) implies that

$$\frac{\partial^2 V}{\partial x^2}(0) = X,$$

where the matrix X is the solution to the ARE (5.9) such that the matrices A_l^* and $A_l^{u^*}$ in (5.10) and (5.11) are asymptotically stable. However, these matrices are the linearization of the vector fields $A^*(x)$ and $A^{\mathbf{u}^*}(x)$ in (5.6) and (5.7) respectively, arround $x=0$, i.e., $A_l^* = \frac{\partial A^*}{\partial x}(0)$, $A_l^{u^*} = \frac{\partial A^{\mathbf{u}^*}}{\partial x}(0)$. Therefore, we conclude that $A^*(x)$ and $A^{\mathbf{u}^*}(x)$ are locally exponentially stable. To see that $V \geq 0$ locally, rewrite the PDE (5.5) as

$$\begin{aligned} &\nabla_x V \cdot A^{\mathbf{u}^*}(x) + \tfrac{1}{2}[\nabla_x V(x) B_1(x) - C_u(x)'\bar{S}(x)']\bar{Q}^{-1}(x) \\ &\qquad [B_1(x)'\nabla_x V(x)' - \bar{S}(x) C^{\mathbf{u}^*}(x)] - C^{\mathbf{u}^*}(x)' R C^{\mathbf{u}^*}(x) = 0, \end{aligned} \tag{5.15}$$

with $C^{\mathbf{u}^*}(x) = C_1(x) + D_{12}(x)\mathbf{u}^*(x)$. Since $\bar{Q}(x) > 0$ for all x, and $R \leq 0$, we have $\nabla_x V \cdot A^{\mathbf{u}^*}(x) \leq 0$. Integrating this inequality along the trajectory produced by $A^{\mathbf{u}^*}$, with $x(0) = x_0$, yields

$$V(x(t)) - V(x_0) \leq 0,$$

provided $x(\cdot)$ stays in W^0. Since $A^{\mathbf{u}^*}(x)$ is locally exponentially stable, choosing x_0 in the domain of attraction for $A^{\mathbf{u}^*}$ results in a trajectory $x(t)$ that goes to 0 as $t \to \infty$. Thus

$$0 = V(0) = V(\lim_{t\to\infty} x(t)) = \lim_{t\to\infty} V(x(t)) \leq V(x_0).$$

Finally, we shall show that under the policy $\mathbf{u}^*(x)$ in (5.8), the closed loop system $\Sigma^{\mathbf{u}^*}$ (2.1) is locally dissipative with respect to the supply rate $r_q(z,w)$. Let W denote the domain on which the function V is defined and is nonnegative. By rewriting the PDE (5.5) as in (5.15) we see that under the policy $\mathbf{u}^*(x)$ the following inequality holds

$$\begin{aligned}
-\int_0^T & r(z(s), w(s))ds \\
&= -\tfrac{1}{2}\int_0^T ([w(s) - w^*(s)]'\bar{Q}[w(s) - w^*(s)] + 2dV/ds)ds \\
&= -\tfrac{1}{2}\int_0^T [w(s) - w^*(s)]'\bar{Q}[w(s) - w^*(s)]ds + V(x(0)) - V(x(T)) \\
&\leq -\tfrac{1}{2}\int_0^T [w(s) - w^*(s)]'\bar{Q}[w(s) - w^*(s)]ds + V(x(0)),
\end{aligned}$$

where $w^*(s) = w^*(x(s)) = \bar{Q}^{-1}(x)[B_1(x)'\nabla_x V(x)' - \bar{S}(x)C^{\mathbf{u}^*}(x)]$, for all $x(0) \in W$, and for all $T \geq 0$, $w \in L_2([0,T])$ such that $x(\cdot)$ does not leave W. This shows the (local) dissipativity of $\Sigma^{\mathbf{u}^*}$ with respect to the supply rate $r_q(z,w)$ in (5.2), and completes the proof. $\square$

6 A Robust Stabilization Problem

In this section we consider an application of the synthesis theory developed in the previous sections to a stabilization problem for a class of uncertain nonlinear systems. This class is described by an interconnection of a fixed (known) nominal system and an unknown system possesing a certain dissipativity property (see Figure 1 below). This property may be boundedness of the H_∞ norm [13], [2], [19], positive realness [15], or sector boundedness [14], [26].

The uncertain system, denoted by $\Sigma_{\mathcal{G},\Delta}$, consists of a (known) nominal system $\mathcal{G}$, where

$$\mathcal{G} : \begin{cases} \dot{x}(t) &= A(x(t)) + B_1(x(t))w(t) + B_2(x(t))u(t),\ x(0) = x_0, \\ z(t) &= C_1(x(t)) + D_{11}(x(t))w(t) + D_{12}(x)u(t), \end{cases} \tag{6.1}$$

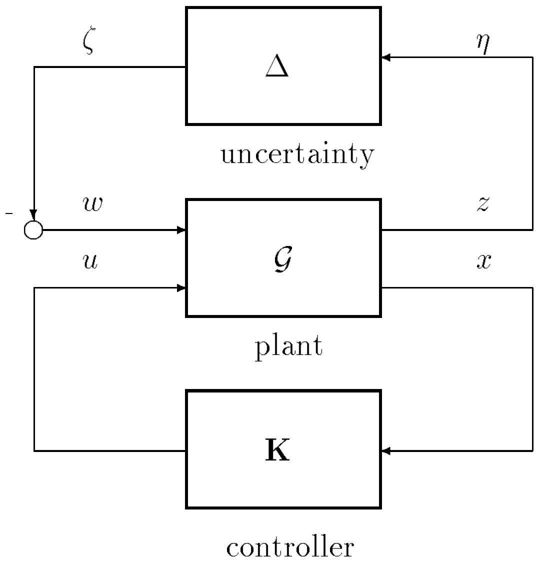

Figure 1. Robust stabilization problem.

with $x \in \mathbf{R}^n,\ w \in \mathbf{R}^d,\ u \in \mathbf{R}^m,\ z \in \mathbf{R}^q$, and an unknown system $\Delta \in \mathcal{D}$

$$\Delta : \left\{ \begin{array}{rcl} \dot{\xi}(t) & = & F_\Delta(\xi(t)) + G_\Delta(\xi(t))\eta(t),\ \xi(0) = \xi_0, \\ \\ \zeta(t) & = & H_\Delta(\xi(t)) + J_\Delta(\xi(t))\eta(t). \end{array} \right. \tag{6.2}$$

with $\xi \in \mathbf{R}^{n_\Delta},\ \ \eta \in \mathbf{R}^q,\ \ \zeta \in \mathbf{R}^d$, in which $\mathcal{D}$ denotes the class of uncertainties. The dimension of the state space realization of Δ, n_Δ may be arbitrary, but finite. Thus, the input-output map of each system $\Delta \in \mathcal{D}$ is completely described by the maps $F_\Delta : \mathbf{R}^{n_\Delta} \to \mathbf{R}^{n_\Delta},\ G_\Delta : \mathbf{R}^{n_\Delta} \to \mathbf{R}^{n_\Delta \times q},\ H_\Delta : \mathbf{R}^{n_\Delta} \to \mathbf{R}^d$ and $J_\Delta : \mathbf{R}^{n_\Delta} \to \mathbf{R}^{d\times q}$. We assume that for each $\Delta \in \mathcal{D}$, the functions $F_\Delta,\ G_\Delta,\ H_\Delta,\ J_\Delta$ satisfy the regularity assumptions in §2, with $F_\Delta(0) = 0,\ H_\Delta(0) = 0$. To ensure the wellposedness of the interconnected systems $\Sigma_{\mathcal{G},\Delta}$, we require that the matrices

$$I + J_\Delta(\xi)D_{11}(x), \quad I + D_{11}(x)J_\Delta(\xi) \tag{6.3}$$

be nonsingular for all $x,\ \xi$.

We consider the following class of uncertainties.

Definition 6.1 *The class of* admissible uncertainties $\mathcal{D}$ *is the set of all systems described in (6.2), which are dissipative with respect to the supply rate $r_\mathcal{D}(\zeta,\eta)$ possesing positive definite and smooth (i.e. C^1) storage functions V_Δ.*

To stabilize the uncertain system $\Sigma_{\mathcal{G},\Delta}$ for all $\Delta \in \mathcal{D}$, we select an appropriate supply rate $r_\mathcal{G}(z,w)$, and attempt to find a controller $u = K(x) \in \mathcal{S}$ to attain dissipativity of the closed loop $\mathcal{G}^u$ with respect to $r_\mathcal{G}(z,w)$.

Theorem 6.2 *Suppose there exist a supply rate $r_{\mathcal{G}}(z,w)$, and a constant $\alpha > 0$ such that the following condition holds*

$$r_{\mathcal{G}}(z,w) + \alpha r_{\mathcal{D}}(-w,z)) \leq 0, \tag{6.4}$$

for all z and w. Assume there exists a positive definite function $V_{\mathcal{G}} \in C^1$ solving the following PDI

$$\inf_{u\in\mathbf{R}^m} \sup_{w\in\mathbf{R}^d}\{\nabla_x V_{\mathcal{G}}(x)\cdot(A(x)+B_1(x)w+B_2(x)u) - r_{\mathcal{G}}(h_1(x,w,u),w)\} \leq 0, \tag{6.5}$$

where $h_1(x,w,u) = C_1(x) + D_{11}(x)w + D_{12}(x)u$. Then the control $\mathbf{u}^*(x)$ which attains minimum on the LHS of the PDI (6.5) will stabilize (in the sense of Lyapunov) the uncertain system $\Sigma_{\mathcal{G},\Delta}$ for all $\Delta \in \mathcal{D}$. If, furthermore, we have: (i) the condition (6.4) holds in the strict sense, i.e.

$$r_{\mathcal{G}}(z,w) + \alpha r_{\mathcal{D}}(-w,z)) < 0,$$

for all $(z,w) \neq (0,0)$ with $r_{\mathcal{G}}(0,0) + \alpha r_{\mathcal{D}}(0,0)) = 0$, and (ii) the systems $\mathcal{G}^{\mathbf{u}^*}$ and each of $\Delta \in \mathcal{D}$ are zero-state detectable, then the control $\mathbf{u}^*(x)$ will asymptotically stabilize the uncertain system $\Sigma_{\mathcal{G},\Delta}$ for all $\Delta \in \mathcal{D}$.

Proof. Suppose $\mathbf{u}^*(x)$ attains minimum on the LHS of PDI (6.5). Then with $u = \mathbf{u}^*$, the PDI (6.5) implies, for any w,

$$\begin{aligned} \frac{dV_{\mathcal{G}}}{dt} &= \nabla_x V_{\mathcal{G}}(x)\cdot(A_{\mathbf{u}^*}(x) + B_1(x)w) \\ &\leq r_{\mathcal{G}}(h_1(x,w,\mathbf{u}^*(x)),w), \end{aligned} \tag{6.6}$$

where $A_{\mathbf{u}^*}(x) = A(x) + B_2(x)\mathbf{u}^*(x)$. Since each $\Delta \in \mathcal{D}$ is dissipative with respect to $r_{\mathcal{D}}(\zeta,\eta)$, Theorem 3.2 implies, for any η,

$$\begin{aligned} \alpha\frac{dV_{\Delta}}{dt} &= \alpha\nabla_\xi V_\Delta(\xi)\cdot(F_\Delta(\xi) + G_\Delta(\xi)\eta) \\ &\leq \alpha r_\Delta(H_{1\Delta}(\xi,\eta),\eta), \end{aligned} \tag{6.7}$$

where $H_{1\Delta}(\xi,\eta) = H_\Delta(\xi) + J_\Delta(\xi)\eta$. Now, define the function

$$V_{\mathcal{G},\Delta}(x,\xi) = V_{\mathcal{G}}(x) + \alpha V_\Delta(\xi).$$

Clearly $V_{\mathcal{G},\Delta}$ is positive definite, i.e., $V_{\mathcal{G},\Delta}(x,\xi) > 0$ for all $(x,\xi) \neq (0,0)$, with $V_{\mathcal{G},\Delta}(0,0) = 0$. By evaluating $\frac{dV_{\mathcal{G},\Delta}}{dt}$ along the trajectory of $(x(\cdot),\xi(\cdot))$ we get

$$\begin{aligned} \frac{dV_{\mathcal{G},\Delta}}{dt} &= \frac{dV_{\mathcal{G}}}{dt} + \alpha\frac{dV_\Delta}{dt} \\ &= \nabla_x V_{\mathcal{G}}(x)\cdot(A_{\mathbf{u}^*}(x) + B_1(x)w) + \alpha\nabla_\xi V_\Delta(\xi)\cdot(F_\Delta(\xi) + G_\Delta(\xi)\eta) \\ &\leq r_G(h_1(x,w,\mathbf{u}^*(x)),w) + \alpha r_\Delta(\xi,\eta) \\ &\leq 0. \end{aligned}$$

Therefore $V_{\mathcal{G},\Delta}(x,\xi)$ is a Lyapunov function for the uncertain system $\Sigma_{\mathcal{G},\Delta}$, and hence, $\Sigma_{\mathcal{G},\Delta}$ is Lyapunov stable for all $\Delta \in \mathcal{D}$.

Suppose that, instead of the condition (6.4), we have

$$r_{\mathcal{G}}(z,w) + \alpha r_{\mathcal{D}}(-w,z) < 0,$$

for all $(z,w) \neq (0,0)$, with $r_{\mathcal{G}}(0,0) + \alpha r_{\mathcal{D}}(0,0) = 0$. Then, by the La Salle invariance principle, $(x(\cdot),\xi(\cdot))$ approaches the set

$$\mathcal{N} = \{(x,\xi) \ : \ h_1(x,0,\mathbf{u}^*(x)) = 0, \text{ and } H_{1\Delta}(\xi,0) = 0\}$$

asymptotically. Therefore, by the zero-state detectability of $\mathcal{G}^{\mathbf{u}^*}$ and each of $\Delta \in \mathcal{D}$, $(x(t),\ \xi(t))$ goes to $(0,0)$ as $t \to \infty$, and thus the uncertain system $\Sigma_{\mathcal{G},\Delta}$ is asymptotically stable for all $\Delta \in \mathcal{D}$ (see also related results in [19], [16], [17],). □

Acknowledgment. This work was supported by the Co-operative Research Centre for Robust and Adaptive Systems ($(\mathrm{CR})^2$ASys), Australia. The authors wish to thank Dr. Ian Petersen for some insightful discussions.

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A Appendix

Definition of viscosity solutions

We recall the definition of weak (viscosity) solutions for nonlinear first order partial differential inequalities ([12], [25]). In what follows, $H : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ is locally bounded above and lower semicontinuous

A locally bounded function V is said to satisfy the PDI

$$H(x, \nabla_x V) \;\leq\; 0 \;\text{ in } \mathbf{R}^n$$

in the weak (viscosity) sense if for every $\phi \in C^1(\mathbf{R}^n)$ and any local minimum $x_0 \in \mathbf{R}^n$ of $V_* - \phi$ one has

$$H(x_0, \nabla_x \phi(x_0)) \ \leq \ 0,$$

where $V_*(x) \triangleq \liminf_{y\to x} V(y)$ is the lower semi continuous envelope of V.

A locally bounded function V is called a viscosity subsolution (respectively supersolution) of the PDE

$$H(x_0, \nabla_x \phi(x_0)) \ = \ 0.$$

if for every $\phi \in C^1(\mathbf{R}^n)$ and any local maximum (respectively minimum) $x_0 \in \mathbf{R}^n$ of $V^* - \phi$ (respectively $V_* - \phi$) one has

$$H(x_0, \nabla_x \phi(x_0)) \ \geq \ 0 \text{ (respectively } \leq 0).$$

Here, $V^*(x) \triangleq \limsup_{y\to x} V(y)$ is the upper semicontinuous envelope of V.

Proof that V is a viscosity subsolution

We shall now show that the value function V defined by

$$V(x) = \inf_{K\in\mathcal{S}} \sup_{T\geq 0, w\in L_2([0,T],\mathbf{R}^d)} \{-\tfrac{1}{2}\int_0^T r_q(h_1(x(t), w(t), K[x](t)), w(t))dt\} \qquad \text{(A.1)}$$

where $r_q(z,w) = \frac{1}{2}(w'Qw + 2w'Sz + z'Rz)$, and $h_1(x,w,u) = C_1(x) + D_{11}(x)w + D_{12}(x)u$ is a viscosity subsolution of the PDE

$$H(x,p) = 0, \qquad \text{(A.2)}$$

where

$$H(x,p) = \inf_{u\in\mathbf{R}^m} \sup_{w\in\mathbf{R}^d} \{p' \cdot (A(x) + B_1(x)w + B_2(x)u) - \tfrac{1}{2}(w'Qw + 2w'Sh_1(x,w,u)$$

$$+ h_1(x,w,u)'Rh_1(x,w,u))\}.$$

We follow the techniques employed in [25]. The result also holds when the set $\mathcal{S}$ is a singleton such as in the open loop problem in §3.

Consider the following monotonically increasing change of parameter

$$\tau_{u,w}(t) = \int_0^t (1 + |u(s)|^2 + |w(s)|^2)ds.$$

Suppose that for all $u_1, u_2 \in L_2([0,\infty),\mathbf{R}^m)$, $w_1, w_2 \in L_2([0,\infty),\mathbf{R}^d)$, we define $u \in L_2([0,\infty),\mathbf{R}^m)$ as

$$u(t) = \begin{cases} u_1(t), & [0,\ t_{u_1,w_1}(\tau_1)), \\ u_2(t), & [t_{u_1,w_1}(\tau_1)\ \infty), \end{cases}$$

and similarly for $w \in L_2([0,\infty), \mathbf{R}^d)$. Then we have

$$u(t_{u,w}(\tau)) = \begin{cases} u_1(t_{u_1,w_1}(\tau)), & [0,\ \tau_1), \\ u_2(t_{u_2,w_2}(\tau - \tau_1)), & [\tau_1,\ \infty). \end{cases}$$

The same property holds similarly for w ([25]).

Consider now the reparametrized state $y(\tau) = x(t(\tau))$. The dynamics of y is given by

$$\begin{aligned} dy/d\tau &= (A(y(\tau)) + B_1(y(\tau))w(\tau) + B_2(y(\tau))u(\tau)) \cdot dt/d\tau \\ &= (A(y(\tau)) + B_1(y(\tau))w(\tau) + B_2(y(\tau))u(\tau)) \cdot (1 + |u(\tau)|^2 + |w(\tau)|^2)^{-1} \\ &= \bar{f}(y(\tau), u(\tau), w(\tau)), \end{aligned} \tag{A.3}$$

with initial condition $y(0) = x(t(0)) = x(0) = x$. Moreover, in terms of τ we have

$$V(x) = \inf_{K \in \mathcal{S}} \sup_{\tau \geq 0, w \in L_2([0,\tau], \mathbf{R}^d)} \{-\tfrac{1}{2}\int_0^\tau \bar{r}(h_1(y(\sigma), w(\sigma), K[y](\sigma)), w(\sigma)) d\sigma : y(0) = x\}, \tag{A.4}$$

where

$$\begin{aligned} \bar{r}(h_1(x,w,u),w) = \tfrac{1}{2}(w'Qw &+ 2w'Sh_1(x,w,u) \\ &+ h_1(x,w,u)'Rh_1(x,w,u)) \cdot (1 + |u|^2 + |w|^2)^{-1}. \end{aligned}$$

The corresponding PDE is given by

$$\mathcal{H}(x,p) = 0 \tag{A.5}$$

where

$$\mathcal{H}(x,p) = \inf_{u \in \mathbf{R}^m} \sup_{w \in \mathbf{R}^d} \{p' \cdot \bar{f}(x,u,w) - \bar{r}(h_1(x,w,u),w)\}.$$

The functions $\bar{f}$ and $\bar{r}$ enjoy the following regularity

$$\begin{aligned} &|\bar{f}(x,u,w) - \bar{f}(x_0,u,w)| \leq L_\rho |x - x_0|, \\ &|\bar{r}(h_1(x,w,u),w) - \bar{r}(h_1(x_0,w,u),w)| \leq L_\rho |x - x_0|, \end{aligned} \tag{A.6}$$

for all $x,\ x_0 \in B(0,\rho),\ \rho > 0$, for all $u \in \mathbf{R}^m$, for all $w \in \mathbf{R}^d$, for some constant L_ρ depending only on ρ. Furthermore, the trajectory $y(\cdot)$ enjoys the following regularity (see Remark 3.2 in [25])

$$|y(s) - x| \leq \gamma(\tau), \tag{A.7}$$

for all $x \in B(0,\rho),\ s \in [0,\ \tau]$, where $\gamma > 0$ is nondecreasing function which is continuous at 0, and $\gamma(0) = 0$. In particular, $|y(\tau) - x| \leq o(1)$ as $\tau \to 0$, uniformly in u, w for all $x \in B(0,\rho)$.

Note that since $-r(z,0) \geq 0$ for all z, we have

$$-\bar{r}(z,0) \geq 0, \quad \forall z \in \mathbf{R}^q.$$

This implies that, for each $K \in \mathcal{S}$, the finite time horizon cost function $J^K(x,\tau)$ defined by

$$J^K(x,\tau) = \sup_{w \in L_2([0,\tau],\mathbf{R}^d)} \{-\int_0^\tau \bar{r}(h_1(y(\sigma),w(\sigma),K[y](\sigma)),w(\sigma))d\sigma\}$$

is monotonically increasing in τ. In particular, we have

$$\begin{aligned} J^K(x) &= \sup_{\tau\geq 0, w \in L_2([0,\tau],\mathbf{R}^d)} \{-\int_0^\tau \bar{r}(h_1(y(\sigma),w(\sigma),K[y](\sigma)),w(\sigma))d\sigma\} \\ &= \sup_{\tau \geq 0} J^K(x,\tau) \\ &= \lim_{\tau\to\infty} J^K(x,\tau). \end{aligned} \tag{A.8}$$

We will need the following dynamic programming principle.

Proposition A.1 *The value function V in (A.4) satisfies the dynamic programming principle*

$$V(x) = \inf_{K\in\mathcal{S}} \sup_{w \in L_2([0,\tau],\mathbf{R}^d)} \{-\tfrac{1}{2}\int_0^\tau \bar{r}(h_1(y(\sigma),w(\sigma),K[y](\sigma)),w(\sigma))d\sigma + V(y(\tau))\}, \tag{A.9}$$

for all $\tau \geq 0$.

Proof. ¿From Lemma 3.4 in [25], we have

$$V(x) \geq \inf_{K\in\mathcal{S}} \sup_{w \in L_2([0,\tau],\mathbf{R}^d)} \{-\tfrac{1}{2}\int_0^\tau \bar{r}(h_1(y(\sigma),w(\sigma),K[y](\sigma)),w(\sigma))d\sigma + V(y(\tau))\},$$

for all $\tau \geq 0$. It remains to show that the reverse inequality holds. For fixed $\bar{\tau} \geq 0$, let $W(x)$ denote the RHS of (A.9), i.e.

$$W(x) = \inf_{K\in\mathcal{S}} \sup_{w \in L_2([0,\bar{\tau}],\mathbf{R}^d)} \{-\frac{1}{2}\int_0^{\bar{\tau}} \bar{r}(h_1(y(\sigma),w(\sigma),K[y](\sigma)),w(\sigma))d\sigma + V(y(\tau))\}.$$

By the definition of W, there exists $K^\epsilon \in \mathcal{S}$ such that

$$W(x) \geq \sup_{w \in L_2([0,\bar{\tau}],\mathbf{R}^d)} \{-\frac{1}{2}\int_0^{\bar{\tau}} \bar{r}(h_1(y(\sigma),w(\sigma),K^\epsilon[y](\sigma)),w(\sigma))d\sigma + V(y(\tau))\} - \frac{\epsilon}{2},$$

for all w. Moreover, the definition of V implies there exists $K^{\epsilon\epsilon} \in \mathcal{S}$ such that

$$V(y(\bar{\tau})) \geq \sup_{\tau\geq\bar{\tau}, w \in L_2([\bar{\tau},\tau],\mathbf{R}^d)} \{-\int_{\bar{\tau}}^\tau \bar{r}(h_1(y(\sigma),w(\sigma),K^{\epsilon\epsilon}[y](\sigma)),w(\sigma))d\sigma\} - \frac{\epsilon}{2}.$$

Combining the last two inequalities, and defining the feedback law

$$\bar{K}[y](\tau) = \begin{cases} K^{\epsilon}[y](\tau), & [0,\ \bar{\tau}) \\ \\ K^{\epsilon\epsilon}[y](\tau), & [\bar{\tau},\ \infty) \end{cases}$$

we get

$$W(x) \geq -\frac{1}{2}\int_0^{\tau} \bar{r}(h_1(y(\sigma), w(\sigma), \bar{K}[y](\sigma)), w(\sigma))d\sigma - \epsilon,$$

for all $w \in L_2([0,\tau], \mathbf{R}^d)$, for all $\tau \geq \bar{\tau}$. Thus, using $\bar{K}$, and the monotonicity in (A.8) we have

$$W(x) \geq \sup_{\tau\geq\bar{\tau}, w\in L_2([0,\tau],\mathbf{R}^d)}\{-\tfrac{1}{2}\int_0^{\tau} \bar{r}(h_1(y(\sigma), w(\sigma), \bar{K}[y](\sigma)), w(\sigma))d\sigma\} - \epsilon$$

$$= \sup_{\tau\geq 0, w\in L_2([0,\tau],\mathbf{R}^d)}\{-\tfrac{1}{2}\int_0^{\tau} \bar{r}(h_1(y(\sigma), w(\sigma), \bar{K}[y](\sigma)), w(\sigma))d\sigma\} - \epsilon.$$

Therefore,

$$W(x) \geq \inf_{K\in\mathcal{S}} \sup_{\tau\geq 0, w\in L_2([0,\tau],\mathbf{R}^d)}\{-\tfrac{1}{2}\int_0^{\tau} \bar{r}(h_1(y(\sigma), w(\sigma), K[y](\sigma)), w(\sigma))d\sigma\} - \epsilon,$$

$$= V(x) - \epsilon.$$

Since $\epsilon > 0$ is abritrary, we get the reverse inequality. □

Now we show that the value V in (A.4) is a viscosity subsolution of the PDE (A.5). Let $x_0 \in \text{argmax}\{V^* - \phi\}$, $\phi \in C^1$, with $V^*(x_0) = \phi(x_0)$. Assume that V is not a viscosity subsolution of the PDE (A.5), i.e.

$$\mathcal{H}(x_0, \nabla_x\phi(x_0)) < 0.$$

Then, there exists $\bar{u} \in \mathbf{R}^m$, $\sigma > 0$ such that for all $w \in \mathbf{R}^d$ we have

$$\nabla_x\phi(x_0)\cdot \bar{f}(x_0, \bar{u}, w) - \bar{r}(h_1(x_0, w, \bar{u}), w) < -\sigma < 0.$$

By the properties of $\bar{f}$, $\bar{r}$ in (A.6) we have,

$$\nabla_x\phi(x)\cdot \bar{f}(x, \bar{u}, w) - \bar{r}(h_1(x, w, \bar{u}), w) \leq -\epsilon,$$

for all $x \in B(x_0, \epsilon)$, for some $\epsilon > 0$. By the property (A.7), there exists $\tau > 0$ (small enough) such that

$$|y(s) - x| \leq \epsilon/2, \quad 0 \leq s \leq \tau,$$

for all $x \in B(x_0, \epsilon/2)$. This implies

$$\nabla_x\phi(y(s))\cdot \bar{f}(y(s), \bar{u}, w(s)) - \bar{r}(h_1(y(s), w(s), \bar{u}), w(s)) \leq -\epsilon, \tag{A.10}$$

for all $0 \leq s \leq \tau$, for all $w \in L_2([0,\tau], \mathbf{R}^d)$. Defining the constant feedback law $\bar{K}[y](s) = \bar{u}, \quad s \in [0,\ \tau]$ and integrating (A.10) we get

$$\phi(y(\tau)) - \phi(x) - \int_0^\tau \bar{r}(h_1(y(s), w(s), \bar{K}[y](s)), w(s))ds \leq -\epsilon\tau,$$

for $w \in L_2([0,\tau], \mathbf{R}^d)$. Adding $-V^*(x_0) + \phi(x_0) = 0$ to the LHS of the last inequality, and using the property $V(y(\tau)) \leq V^*(y(\tau)) \leq \phi(y(\tau))$ (provided ϵ is small enough), yields

$$-V^*(x_0) + \phi(x_0) - \phi(x) + V(y(\tau)) - \int_0^\tau \bar{r}(h_1(y(s), w(s), \bar{K}[y](s)), w(s))ds \leq -\epsilon\tau,$$

for $w \in L_2([0,\tau], \mathbf{R}^d)$. Thus,

$$V^*(x_0) \geq \epsilon\tau + \phi(x_0) - \phi(x) + \int_0^\tau \bar{r}(h_1(y(s), w(s), \bar{K}[y](s)), w(s))ds + V(y(\tau)),$$

for $w \in L_2([0,\tau], \mathbf{R}^d)$. Taking supremum over w, using $\bar{K}$, we get

$$V^*(x_0) \geq \epsilon\tau + \phi(x_0) - \phi(x) + \sup_{w \in L_2([0,\tau],\mathbf{R}^d)} \{\int_0^\tau \bar{r}(h_1(y(s), w(s), \bar{K}[y](s)), w(s))ds + V(y(\tau))\}.$$

Taking infimum over $K \in \mathcal{S}$, and using Proposition A.1 yields

$$\begin{aligned} V^*(x_0) &\geq \epsilon\tau + \phi(x_0) - \phi(x) \\ &+ \ \inf\nolimits_{K \in \mathcal{S}} \sup\nolimits_{w \in L_2([0,\tau],\mathbf{R}^d)} \{\int_0^\tau \bar{r}(h_1(y(s), w(s), K[y](s)), w(s))ds + V(y(\tau))\} \\ &= \epsilon\tau + \phi(x_0) - \phi(x) + V(x). \end{aligned}$$

Next, consider the sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x_0, \quad \lim_{n\to\infty} V(x_n) = V^*(x_0)$. There exists large enough N such that $x_n \in B(x_0, \epsilon/2)$ for all $n \geq N$. Setting $x(0) = x_n,\ n \geq N$, we get

$$V^*(x_0) \geq \epsilon\tau + \phi(x_0) - \phi(x_n) + V(x_n).$$

As $n \to \infty$ we obtain

$$V^*(x_0) \geq \epsilon\tau + V^*(x_0),$$

which is a contradiction. Therefore, we must have

$$\mathcal{H}(x_0, \nabla_x \phi(x_0)) \geq 0. \tag{A.11}$$

Finally, it follows from the proof of Theorem 4.9 of [25] that (A.11) implies

$$H(x_0, \nabla_x \phi(x_0)) \geq 0.$$

This completes the proof.