On the Stability of the Information State System

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Abstract

The purpose of this paper is to present some preliminary results on the stability of the information state system. The information state system underlies the (infinite dimensional) dynamics of an H_∞ controller for a nonlinear system. Thus it is important to understand its stability and the structure of its equilibrium points. We analyse the important case corresponding to the mixed sensitivity problem. We prove the existence of an equilibrium information state, convergence under very general conditions to such an equilibrium state p_e and uniqueness of this state (up to an irrelevant constant). In this case the equilibrium p_e is usually singular in the sense that it takes on the value $-\infty$ except on a low dimensional subset of its domain.

This meshes with the article [9] which analysed the effect of using p_e to initialize the information state controller and gave explicit formulas which in many cases produce a dramatic reduction in the amount of computation required to implement the controller. What this article suggests is that indeed p_e is the only equilibrium initialization possible.

Key words: Nonlinear H_∞ control, information states, infinite dimensional dynamical systems, stability.

1 Introduction

The information state PDE gives the dynamics of the one of the main candidates for a H_∞ controller called the *central controller*. The nature of the central controller dynamics depends not just on the information state PDE but on the function p_0 used as an initial condition for this PDE, [8,9,10]. We denote by $K^*_{p_0}$ the central controller with initial condition p_0. The effect of initialization is very important and not fully understood. This paper gives evidence that a special equilibrium or steady state information state p_e gives the "correct" choice of central controller initialization.

A key issue is that in general the equilibrium information state p_e is extended real valued, i.e., it can take on the value $-\infty$ on a large subset of the state space. Such cases are termed *singular*. This presents both technical difficulties as well as offering the potential to dramatically reduce the complexity of the controller, [9,10]. This latter virtue is very important from the practical point of view and is developed further in [10]; indeed in one case it turns out that the dynamics are finite dimensional. In *non-singular* cases, such complexity reduction is not generally possible; however, theory for these cases is much more complete.

The purpose of this paper is to present some results concerning the stability and asymptotics of the information state system (which underlies the central controller) which in turn bears on how one should

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initialize this PDE. We do not give a fully motivated discussion of central controllers, but merely work with the key formulas for defining the dynamics of this controller. See [8,9,11,12] for more details. A complete, detailed treatment will be provided in [10].

2 The Information State System

We consider nonlinear plants G of the form

$$G : \begin{cases} \dot{x} &= A(x) + B_1(x)w + B_2(x)u \\ z &= C_1(x) + D_{12}(x)u \\ y &= C_2(x) + D_{21}(x)w \end{cases} \tag{1}$$

Here, $x(t) \in \mathbf{R}^n$ denotes the state of the system, and is not in general directly measurable; instead an output $y(t) \in \mathbf{R}^p$ is observed. The additional output quantity $z(t) \in \mathbf{R}^r$ is a performance measure, depending on the particular problem at hand. The control input is $u(t) \in \mathbf{R}^m$, while $w(t) \in \mathbf{R}^s$ is regarded as an opposing disturbance input. We assume that all of the functions appearing in (1) are as smooth with bounded first and second derivatives, that B_1, B_2, D_{12} and D_{21} are *independent of* x, and that zero is an equilibrium: $A(0) = 0$, $C_1(0) = 0$ and $C_2(0) = 0$.

In this paper we consider the *one block problem*, defined by

$$D_{12}(x) = I_m \quad (r = m), \quad \text{and } D_{21}(x) = I_p \quad (s = p). \tag{$1-BLK$}$$

The results of this paper hold with only trivial modification for the *two block* problem, while the structure of the four block problem, see [10], is different. The two block problem relaxes the constraint $D_{21}(x) = I_p$ $(s = p)$ and in control problems arises from the mixed sensitivity problem.

For fixed $u, y \in L_{2,e} = \cup_{T \geq 0} L_2[0,T]$, the *information state* $p_t \in \mathcal{X}_e$ is defined by

$$p_t(x) = p_0(\xi(0)) + \tfrac{1}{2}\int_0^t [|C_1(\xi(s)) + u(s)|^2 - \gamma^2 |y(s) - C_2(\xi(s))|^2]\, ds \tag{2}$$

where $\xi(s)$, $0 \leq s \leq t$, satisfies the "reversed-arrow" dynamics

$$\dot{\xi} = A(\xi) + B_1(\xi)(y - C_2(\xi)) + B_2(\xi)u \tag{3}$$

with terminal condition $\xi(t) = x$. (We shall use the semigroup notation

$$\Phi_{s,t}^{u,y}(x)$$

to denote the solution $\xi(s)$ just defined.) The system (3) has inputs u, y, and plays a major role in the behavior of the information state p_t. In particular, the stability properties of the vector field

$$A^{\times} \triangleq A - B_1 C_2$$

is crucial.

The dynamics for p_t is a partial differential equation: for fixed $u \in L_{2,e}$ and $y \in L_{2,e}$ we have

$$\dot{p}_t = F(p_t, u(t), y(t)), \tag{4}$$

where $F(p, u, y)$ is the differential operator

$$F(p, u, y) = -\nabla_x p \cdot (A + B_1(y - C_2) + B_2 u) + \tfrac{1}{2}|C_1 + u|^2 - \gamma^2 \tfrac{1}{2}|y - C_2|^2. \tag{5}$$

If p_0 is smooth, then p_t will be smooth for all $t \geq 0$; otherwise, (4) can be interpreted in the sense of formula (2) (or in the viscosity sense).

The system (4) can be viewed as an infinite dimensional system with inputs u, y, and it defines the underlying state of the central H_∞ controller, see [11,10].

Our goal is to investigate the asymptotic behavior of p_t as $t \to \infty$. We will show essentially that

$$\lim_{t\to\infty} p_t = p_e + \psi, \tag{6}$$

where $\psi \in \mathbf{R}$ is a constant (depending on u, y) and p_e is a (formal) solution of the *steady state information state equation*

$$0 = F(p_e, 0, 0); \tag{7}$$

i.e.

$$0 = -\nabla_x p_e \cdot (A - B_1 C_2) + \tfrac{1}{2}|C_1|^2 - \gamma^2 \tfrac{1}{2}|C_2|^2. \tag{8}$$

Precise statements will be given below in (10).

As we shall see, the function p_e is often *singular*, i.e. extended real valued ($p_e(x) = -\infty$ for some x). Indeed, we shall make use of the functions of the form

$$\delta_M(x) = \begin{cases} 0 & \text{if } x \in M \\ -\infty & \text{if } x \notin M \end{cases} \tag{9}$$

where $M \subset \mathbf{R}^n$ (typically a manifold or a point).

The function p_e is called a *stationary* or *equilibrium solution* of (8) if it satisfies

$$p_e(x) = p_e(\Phi^{0,0}_{0,T}(x)) + \tfrac{1}{2}\int_0^T [|C_1(\Phi^{0,0}_{s,T}(x))|^2 - \gamma^2 |C_2(\Phi^{0,0}_{s,T}(x))|^2] ds \tag{10}$$

the integral form of (8).

3 Motivation: Linear Systems

To motivate and illustrate the more general nonlinear results below, we consider the case of linear systems:

$$A(x) = Ax, \quad C_1(x) = C_1 x, \quad C_2(x) = C_2 x.$$

The vector field $A^\times$ is now a linear vector field

$$A^\times(x) = (A - B_1 C_2)x,$$

and we write $A^\times = A - B_1 C_2$ for this matrix. We shall assume that $A^\times$ is *hyperbolic*, meaning that all eigenvalues have strictly negative or strictly positive real parts (no eigenvalues have zero real part).

The information state is given explicitly by

$$p_t(x) = -\gamma^2 \tfrac{1}{2}(x - \hat{x}(t))' Y(t)^{-1} (x - \hat{x}(t)) + \phi(t), \tag{11}$$

(assuming Y is invertible) where

$$\begin{aligned} \dot{\hat{x}} &= \Gamma A - B_1 C_2 + Y(\gamma^{-2} C_1' C_1 - C_2' C_2)\Gamma \hat{x} \\ &\quad + (B_1 + Y C_2')y + (B_2 + \gamma^{-2} Y C_1')u, \end{aligned} \tag{12}$$

$$\dot{Y} = (A - B_1 C_2)Y + Y(A - B_1 C_2)' + Y(\gamma^{-2} C_1' C_1 - C_2' E_2^{-1} C_2)Y \tag{13}$$

$$\dot{\phi} = \tfrac{1}{2}|C_1 \hat{x} + u|^2 - \gamma^2 \tfrac{1}{2}|y - C_2 \hat{x}|^2. \tag{14}$$

This can be readily verified by differentiation and substitution in (4).

The stationary version of the Riccati differential equation (13) is the algebraic Riccati equation

$$0 = (A - B_1 C_2)Y_e + Y_e(A - B_1 C_2)' + Y_e(\gamma^{-2}C_1'C_1 - C_2'C_2)Y_e. \tag{15}$$

One of the (three) necessary and sufficient conditions for the solvability of the H_∞ control problem is the existence of a *stabilizing* solution $Y_e \geq 0$ of (15), meaning that the matrix

$$A - B_1 C_2 + Y_e(\gamma^{-2}C_1'C_1 - C_2'C_2)$$

has all eigenvalues with strictly negative real parts.

In the nonlinear case (and one could argue in the linear case) Y^{-1} rather than Y is what occurs naturally. In the linear case it is easy to change variables to get everything in terms of Y, but in seriously nonlinear cases such a change of varibles appears impossible. Thus one is stuck with $Y_I(t) = Y(t)^{-1}$, which if $Y(t) > 0$, satisfies the Riccati differential equation

$$\dot{Y}_I = -Y_I(A - B_1 C_2) - (A - B_1 C_2)'Y_I - (\gamma^{-2}C_1'C_1 - C_2'C_2), \tag{16}$$

with corresponding steady state Riccati equation

$$0 = -Y_{Ie}(A - B_1 C_2) - (A - B_1 C_2)'Y_{Ie} - (\gamma^{-2}C_1'C_1 - C_2'E_2^{-1}C_2). \tag{17}$$

to which one seeks an *antistablizing solution* Y_{Ie}. Equation (17) corresponds to (8).

A crucial issue is whether Y_e is invertible and how to make sense of it when Y_e is not.

There are three cases to consider: (i) $A^\times$ asymptotically stable; (ii) $A^\times$ asymptotically antistable (i.e. $-A^\times$ asymptotically stable); and (iii) $A^\times$ hyperbolic. In cases (i) and (iii), Y_e is not invertible, while in case (ii) Y_e is invertible. This follows from [13, Theorem 3.4]. Of course, case (iii) includes (i) and (ii) as special cases.

In case (ii), $Y_e > 0$ and (15) implies that the matrix $A - B_1 C_2 + Y_e(\gamma^{-2}C_1'C_1 - C_2'C_2)$ is similar to the matrix $-(A - B_1 C_2)'$.

If $A^\times$ is hyperbolic, there exist coordinates such that $A^\times$ is equivalent to the block diagonal matrix

$$A^\times \asymp \begin{pmatrix} A_s & 0 \\ 0 & A_{as} \end{pmatrix},$$

corresponding to a stable invariant subspace $M_s = \{(x_1, \ldots, x_{n_s}, 0, \ldots, 0)\}$, and an antistable invariant subspace $M_{as} = \{(0, \ldots, 0, x_{n_s+1}, \ldots, x_n)\}$. In these coordinates, Y_e takes the form

$$Y_e \asymp \begin{pmatrix} 0 & 0 \\ 0 & Y_{as-e} \end{pmatrix}, \tag{18}$$

with $Y_{as-e} > 0$.

Our interest is in the asymptotics

$$Y(t)^{-1} \to Y_e^{-1} = \text{`}Y_{Ie} \asymp \begin{pmatrix} -\infty & -\infty \\ -\infty & Y_{as-e} \end{pmatrix}',$$

as $t \to \infty$ for various initializations $Y(0) = Y_0 \geq 0$, in the interpretation of this formal expression, and in the subsequent non-linear generalization to (6). Indeed what we find is that the appropriate definition of $(Y_e^{-1}x, x)$ is the singular function

$$p_e(x) = \delta_{M_{as}}(x) + \phi_e(x), \tag{19}$$

and where $\phi_e : M_{as} \to \mathbf{R}$, viz. the quadratic form

$$\phi_e(x_s, x_{as}) = -\gamma^2 \tfrac{1}{2} x_{as}' Y_{as-e}^{-1} x_{as} \tag{20}$$

where $x = (x_s, x_{as})$ give coordinates relative to $\mathbf{R}^n = M_s \oplus M_{as}$. Note ϕ_e is a smooth function on M_{as}.

This has implications for initializing the information state equation, namely, if $Y_0 = Y_e$, and $u \in L_2$ and $y \in L_2$ then $Y(t) = Y_e$ for all $t \geq 0$ and

$$p_t(x) = \delta_{M_{as}}(x - \hat{x}(t)) - \gamma^2 \tfrac{1}{2}(x_{as} - \hat{x}_{as}(t))' Y_{as-e}^{-1}(x_{as} - \hat{x}_{as}(t)) + \phi(t)$$

$$\to \delta_{M_{as}}(x) - \gamma^2 \tfrac{1}{2} x'_{as} Y_{as-e}^{-1} x_{as} + \psi$$

as $t \to \infty$, for some constant ψ.

4 Main Results: Nonlinear Systems

In this section we will show that in general the stable equilibrium solutions of the steady state information state equation have the form (19), where M_{as} is the antistable manifold of the hyperbolic vector field $A^\times$, and also prove that the stability result (6) holds under certain conditions.

We now state the definitions of stability and hyperbolicity of a vector field f (globally Lipschitz continuous) on $\mathbf{R}^n$ we need:

- f is *strongly stable* if f is globally exponentially stable, i.e. for all initial states x_0 of the ODE $\dot{x} = f(x)$ we have
$$|x(t)| \leq C|x_0|e^{-\rho t} \quad \text{for all } t \geq 0, \tag{21}$$
where C and ρ are positive constants.
- f is *strongly antistable* if $-f$ is strongly stable.
- f is *strongly hyperbolic* if there exist global stable M_s and antistable M_{as} manifolds such that $\mathbf{R}^n = T_0 M_s \oplus T_0 M_{as}$ and f restricted to M_s is strongly stable and $-f$ restricted to M_{as} is strongly stable, and further that there exist coordinates such that $\mathbf{R}^n = M_s \oplus M_{as}$.

A simple criteria for strong stability is

$$\xi' \nabla_x f(x) \xi \leq -c|\xi|^2 \qquad (c > 0), \tag{22}$$

in which case a Lyapunov function is $L(x) = \frac{1}{2}|x|^2$ (by the mean value theorem, (22) implies $x' f(x) \leq -c|x|^2$). *For simplicity, we shall assume this criteria when speaking of strong stability/hyperbolicity.*

4.1 The Nonsingular Case

In this subsection we assume that $A^\times$ is strongly antistable. As we shall see, this implies the existence of an equilibrium solution p_e of the steady state equation (8) which is nonsingular (finite for all $x \in \mathbf{R}^n$).

Lemma 4.1 *Assume $A^\times$ is strongly antistable. Then there exists an equilibrium solution $p_e \in C^1(\mathbf{R}^n)$ of (8) such that*

$$-c_1|x|^2 \leq p_e(x) \leq c_2|x|^2 \tag{23}$$

for some constants $c_1, c_2 > 0$, $p_e(0) = 0$, and

$$|\nabla_x p_e(x)| \leq C|x|.$$

Moreover, any other solution $q_e \in C^1(\mathbf{R}^n)$ satisfies $q_e = p_e + c$, for some real number c.

PROOF. Set $p_0 = 0$, $u = 0$ and $y = 0$. Then

$$p_t(x) = \tfrac{1}{2}\int_0^t [|C_1(\Phi^{0,0}_{s,t}(x))|^2 - \gamma^2|C_2(\Phi^{0,0}_{s,t}(x))|^2]\,ds.$$

Then using the estimate (41) with $u = 0$, $y = 0$ gives

$$p_t(x) \geq -\tfrac{1}{2}\int_0^t \gamma^2|C_2(\Phi^{0,0}_{s,t}(x))|^2\,ds \geq -c_1|x|^2.$$

Similarly the reverse inequality is proven. Further, it follows that $p_t(0) = 0$ for all $t \geq 0$.

The estimates in Lemma A.1 imply that $\Phi^{0,0}_{s,t}(x) \to 0$ as $t \to \infty$ and the integrand in the integral defining p_t is integrable on $[0, \infty)$. Hence $\lim_{t\to\infty} p_t$ exists. Call this limit p_e. This limit is uniform on compact sets.

It also follows that $p_e \in C^1(\mathbf{R}^n)$ and p_e solves (8) (details omitted).

If q_e is another solution, set $\phi = p_e - q_e$. Then ϕ satisfies the PDE

$$0 = -\nabla_x\phi \cdot A^\times.$$

Hence

$$\phi(x) = \phi(\Phi^{0,0}_{0,t}(x)) \to \phi(0) = p_e(0) - q_e(0) = -q_e(0)$$

as $t \to \infty$. This completes the proof. □

Theorem 4.2 *Assume $A^\times$ is strongly antistable. Let $u, y \in L_2[0,\infty)$ and $p_0 \in C^1(\mathbf{R}^n)$ satisfy an inequality like (23). Then*

$$\lim_{t\to\infty} p_t = p_e + \psi \tag{24}$$

uniformly on compact subsets, where p_e is given as in Lemma 4.1 (and has bounded second order partial derivatives) and ψ is a real number depending on u, y.

PROOF. Step 1. Write

$$p_t = p_e + \phi_t.$$

Using arbitrary u, y in L_2, we have

$$\begin{aligned}\dot{\phi} &= F(p_e + \phi, u, y) \\ &= -\nabla_x\phi\cdot(A^\times + B_1 y + B_2 u) + \tfrac{1}{2}|u|^2 - \gamma^2\tfrac{1}{2}|y|^2 + (C_1' - \nabla_x p_e B_2)u + (\gamma^2 C_2' - \nabla_x p_e B_1)y\end{aligned} \tag{25}$$

Integrating,

$$\begin{aligned}\phi_t(x) &= \phi_0(\xi(0)) + \int_0^t [\tfrac{1}{2}|u(s)|^2 - \gamma^2\tfrac{1}{2}|y(s)|^2 + (C_1(\xi(s))' - \nabla_x p_e(\xi(s))B_2)u(s) \\ &\qquad + (\gamma^2 C_2(\xi(s))' - \nabla_x p_e(\xi(s))B_1)y(s)]\,ds,\end{aligned} \tag{26}$$

where $\xi(s) = \Phi^{u,y}_{s,t}(x)$.

Step 2. Making use of the estimates in Lemma A.1, we get

$$-c_1(x,u,y) \leq \phi_t(x) \leq c_2(x,u,y) \text{ for all } t \geq 0.$$

Step 3 (gradient tends to zero). Let $\Psi_{s,t}$ denote the (fundamental) solution of

$$\dot{\Psi} = \nabla_x A^\times(\xi(s))\Psi, \quad \Psi_{t,t} = I,$$

where $\xi(s) = \Phi^{u,y}_{s,t}(x)$. Then as in Lemma A.1, we have

$$|\Psi_{s,t}| \leq Ce^{-c(t-s)}.$$

Using the assumed smoothness of $\phi_s(x)$, following [5, Lemma 5.3] we have the following representation

$$\nabla_x \phi_t(x) = \nabla_x \phi_0(\xi(0))\Psi_{0,t} + \int_0^t \ell(\xi(s), u(s), y(s))\Psi_{s,t}\, ds, \tag{27}$$

where $\xi(s) = \Phi^{u,y}_{s,t}(x)$, and

$$\begin{aligned} \ell(x,u,y) &= \nabla_x\{\tfrac{1}{2}|u|^2 - \gamma^2\tfrac{1}{2}|y|^2 + (C_1' - \nabla_x p_e B_2)u + (\gamma^2 C_2' - \nabla_x p_e B_1)y\} \\ &= \nabla_x\{(C_1' - \nabla_x p_e B_2)u + (\gamma^2 C_2' - \nabla_x p_e B_1)y\}. \end{aligned}$$

Our assumptions imply $|\ell(x,u,y)| \leq C(|u| + |y|)$.

The above estimates and those of Lemma A.1 then imply

$$|\nabla_x \phi_t(x)| \leq C(x,u,y)e^{-ct} + \int_0^t C(|u(s)| + |y(s)|)e^{-c(t-s)}\, ds,$$

and so $\lim_{t\to+\infty} \nabla_x \phi_t(x) = 0$ uniformly on compact sets.

Step 4 (limit of $\phi_t(0)$). Set $x = 0$ in the PDE for ϕ_t to get

$$\dot{\phi}_t(0) = -\nabla_x \phi_t(0) \cdot (B_1 y(t) + B_2 u(t)) + \tfrac{1}{2}|u(t)|^2 - \gamma^2\tfrac{1}{2}|y(t)|^2.$$

Integrating,

$$\phi_t(0) = \phi_0(0) + \int_0^t [-\nabla_x \phi_s(0) \cdot (B_1 y(s) + B_2 u(s)) + \tfrac{1}{2}|u(s)|^2 - \gamma^2\tfrac{1}{2}|y(s)|^2]\, ds$$

The estimates from Step 3 imply that the function $s \mapsto \nabla_x \phi_s(0)$ is in $L_2[0,\infty)$. Therefore the integral converges to a finite number, call it $\psi^{u,y}$.

Step 5 (the limit). Because of the gradient bounds, we now know that $\{\phi_t\}_{t\geq 0}$ is precompact (in the topology of local uniform convergence), and any limit point is a bounded, continuous function. In fact, there is only one limit point and it is the constant $\psi^{u,y}$. Indeed, by the fundamental theorem of calculus, we have

$$\phi_t(x) = \phi_t(0) + \int_0^1 \nabla_x \phi_t(\lambda x) \cdot x\, d\lambda,$$

and so

$$\lim_{t\to\infty} \phi_t(x) = \lim_{t\to\infty} \phi_t(0) + 0 = \psi^{u,y},$$

uniformly on compact sets, which is independent of x. This completes the proof. $\square$

4.2 The Pure Singular Case

We now assume that $A^\times$ is strongly stable. This leads to an equilibrium information state $p_e = \delta_0$ which equals $-\infty$ everywhere except at one point $x = 0$; this case is the opposite extreme to that of the previous section.

Lemma 4.3 *The singular function*

$$p_e = \delta_0 \tag{28}$$

is an equilibrium solution of the steady state information state equation (8).

PROOF. If $p_0 = \delta_{x_0}$, then for any $u, y \in L_{2,e}$

$$p_t = \delta_{\xi(t)} + \phi_t, \tag{29}$$

where

$$\phi_t = \tfrac{1}{2}\int_0^t [|C_1(\xi(s)) + u(s)|^2 - \gamma^2 |y(s) - C_2(\xi(s))|^2]\, ds,$$

and $\xi(\cdot)$ is the solution of (3) with initial condition $\xi(0) = x_0$.

To see this, it follows from the definition (2), that if $p_0 = \delta_{x_0}$, the only value of $\xi(0)$ for which $\delta_{x_0}(\xi(0))$ is finite (i.e. 0) is $\xi(0) = x_0$, with endpoint $x = \xi(t)$, and this yields the cost on the RHS of (29); otherwise $p_t(x)$ equals $-\infty$. So $p_t(x)$ is the sum of $\delta_{\xi(t)}$ and the cost term.

Let $x_0 = 0$, $u = 0$ and $y = 0$. Then we have $\xi(t) = 0$ and hence $p_t = \delta_0$ for all $t \geq 0$. □

Theorem 4.4 *Assume $A^\times$ is strongly stable. Let $u, y \in L_2[0, \infty)$ and $p_0 = p_e$ given by (28). Then*

$$\lim_{t\to\infty} p_t = p_e + \psi \tag{30}$$

where ψ is a real number depending on u, y.

PROOF. The strong stability of $A^\times$ implies

$$|\xi(t)|^2 \leq C \int_0^t e^{-c(t-s)}(|u(s)|^2 + |y(s)|^2) ds$$

which is integrable and tends to 0 as $t \to \infty$. Therefore by (29) we have $p_t \to \delta_0 + \psi$, where ψ is the limiting value of the integral. □

4.3 The General Singular Case

In the general case, we assume $A^\times$ to be strongly hyperbolic. This case is a combination of the preceding two cases.

Lemma 4.5 *Assume $A^\times$ is strongly hyperbolic. Then there exists an equilibrium solution p_e of (8) of the form*

$$p_e = \delta_{M_{as}} + \phi_e \tag{31}$$

such that $\phi_e \in C^1(M_{as})$ and $\phi_e(0) = 0$. Moreover, any other solution of the form $q_e = \delta_{M_{as}} + \psi_e$, where $\psi_e \in C^1(M_{as})$ satisfies $\psi_e = \phi_e + c$, for some real number c.

PROOF. Let $M_0 \subset \mathbf{R}^n$ be a manifold and $\phi_0 \in C(M_0)$. Consider the initial information state $p_0 = \delta_{M_0} + \phi_0$. Then the information state at time $t \geq 0$ is given by

$$p_t = \delta_{M_t} + \phi_t,$$

where $M_t = \Phi^{u,y}_{t,0}(M_0)$ and $\phi_t \in C(M_t)$ is given by

$$\phi_t(x) = \phi_0(\Phi^{u,y}_{0,t}(x)) + \tfrac{1}{2}\int_0^t [|C_1(\Phi^{u,y}_{s,t}(x))|^2 - \gamma^2|C_2(\Phi^{u,y}_{s,t}(x))|^2] ds \text{ for } x \in M_t.$$

Now set $M_0 = M_{as}$, $\phi_0 = 0$, $u = 0$ and $y = 0$. Then by invariance, $M_t = M_{as}$ for all $t \geq 0$. Further, as in Lemma 4.1, we see that $\lim_{t\to\infty} \phi_t(x)$ exists for $x \in M_{as}$; call this limit $\phi_e(x)$. The limit is uniform on compact subsets of M_{as}. Actually, $\phi_e \in C^1(M_{as})$. Write $p_e = \delta_{M_{as}} + \phi_e$.

To see that p_e is an equilibrium solution of (8), note that

$$
\begin{aligned}
p_{t+T}(x) &= \delta_{M_{as}}(x) + \phi_{t+T}(x) \\
&= \delta_{M_{as}}(x) + \phi_t(\Phi^{0,0}_{t,t+T}(x)) + \tfrac{1}{2}\int_t^{t+T} [|C_1(\Phi^{0,0}_{s,t+T}(x))|^2 - \gamma^2|C_2(\Phi^{0,0}_{s,t+T}(x))|^2]ds
\end{aligned}
$$

Fixing $x \in M_{as}$, sending $t \to \infty$ and using time invariance, we see that

$$
\phi_e(x) = \phi_e(\Phi^{0,0}_{0,T}(x)) + \tfrac{1}{2}\int_0^T [|C_1(\Phi^{0,0}_{s,T}(x))|^2 - \gamma^2|C_2(\Phi^{0,0}_{s,T}(x))|^2]ds,
$$

showing the ϕ_e is an equilibrium solution of (8) on M_{as}.

Finally, the uniqueness assertion follows as in Lemma 4.1 since $A^\times$ is strongly antistable on M_{as}. □

We say that the smooth function p_0 on R^n is an *infosubequilibrium* for the system (3) provided there is a $\varepsilon \geq 0$ so that p_0 satisfies the inequality

$$
F(p_0, 0, 0)(x) + \varepsilon|x|^2 \leq 0 \tag{32}
$$

for all $x \in \mathbf{R}^n$, and a constant $C > 0$ such that $|\nabla_x p_0(x)| \leq C|x|$ for all $x \in \mathbf{R}^n$.

Theorem 4.6 *Assume $A^\times$ is strongly hyperbolic. Let $u, y \in L_2[0,\infty)$ have compact support. Then*

$$
\lim_{t\to\infty} p_t = p_e + \psi \tag{33}
$$

pointwise, where p_e is given as in Lemma 4.5 and ψ is a real number depending on u, y.

PROOF. Suppose $[0,T]$ contains the support of u and y. Then for $0 \leq s \leq T$,

$$
\Phi^{u,y}_{s,t}(x) = \begin{cases} \Phi^{u,y}_{s,T}\Phi^{0,0}_{T,t}(x) & \text{if } 0 \leq s \leq T, \\ \Phi^{0,0}_{s,t}(x) & \text{if } T \leq s \leq t \end{cases}.
$$

If $x \in M_{as}$, then $\Phi^{0,0}_{T,t}(x) \to 0$ exponentially as $t \to +\infty$, while if $x \notin M_{as}$, then $|\Phi^{0,0}_{T,t}(x)| \to \infty$ as $t \to +\infty$.

Using (2), we have

$$
\begin{aligned}
p_t(x) = p_0(\Phi^{u,y}_{0,T}\Phi^{0,0}_{T,t}(x)) &+ \tfrac{1}{2}\int_0^T [|C_1(\Phi^{u,y}_{s,T}\Phi^{0,0}_{T,t}(x)) + u(s)|^2 - \gamma^2|y(s) - C_2(\Phi^{u,y}_{s,T}\Phi^{0,0}_{T,t}(x))|^2]\, ds \\
&+ \tfrac{1}{2}\int_T^t [|C_1(\Phi^{0,0}_{s,t}(x))|^2 - \gamma^2|C_2(\Phi^{0,0}_{s,t}(x))|^2]\, ds.
\end{aligned} \tag{34}
$$

Let $x \in M_{as}$. Then the first and second terms in (34) tend to the real number $\psi^{u,y}$ given by

$$
\psi^{u,y} = p_0(\Phi^{u,y}_{0,T}(0)) + \tfrac{1}{2}\int_0^T [|C_1(\Phi^{u,y}_{s,T}(0)) + u(s)|^2 - \gamma^2|y(s) - C_2(\Phi^{u,y}_{s,T}(0))|^2]\, ds, \tag{35}
$$

and the third member of (34) converges to the function

$$
\phi(x) = \lim_{t\to+\infty} \tfrac{1}{2}\int_T^t [|C_1(\Phi^{0,0}_{s,t}(x))|^2 - \gamma^2|C_2(\Phi^{0,0}_{s,t}(x))|^2]\, ds. \tag{36}
$$

Note that this uses time-invariance. Set $\phi^{u,y}(x) = \phi_e(x) + \psi^{u,y}$ for $x \in M_{as}$. Therefore

$$
\lim_{t\to\infty} p_t(x) = \phi^{u,y}(x) \text{ for } x \in M_{as}. \tag{37}
$$

For any $x \in \mathbf{R}^n$, we have $p_t(x) > -\infty$; hence $\liminf_{t\to\infty} p_t(x) \geq -\infty$.

Now suppose $x \notin M_{as}$. Since p_0 is an infosubequilibrium, by integration we have

$$p_0(x) \geq p_0(\Phi_{0,t}^{u,y}(x)) + \tfrac{1}{2}\int_0^t [|C_1(\Phi_{s,t}^{u,y}(x))|^2 - \gamma^2|C_2(\Phi_{s,t}^{u,y}(x))|^2]ds$$

$$+\int_0^t [(\varepsilon-\eta)|\Phi_{s,t}^{u,y}(x)|^2 - \frac{C}{\eta}(|u(s)|^2 + |y(s)|^2)]ds.$$

Here (and again), we make use of estimates of the type $|\nabla_x p_0(\xi)B_1 y| \leq \eta|\xi|^2 + \frac{C}{\eta}|y|^2$, valid for any $\eta > 0$. Now by definition

$$\begin{aligned} p_t(x) &= p_0(\Phi_{0,t}^{u,y}(x)) + \tfrac{1}{2}\int_0^t [|C_1(\Phi_{s,t}^{u,y}(x)) + u(s)|^2 - \gamma^2|y(s) - C_2(\Phi_{s,t}^{u,y}(x))|^2]\, ds \\ &\leq p_0(\Phi_{0,t}^{u,y}(x)) + \tfrac{1}{2}\int_0^t [|C_1(\Phi_{s,t}^{u,y}(x))|^2 - \gamma^2|C_2(\Phi_{s,t}^{u,y}(x))|^2]\, ds \\ &\quad + \int_0^t [\alpha|\Phi_{s,t}^{u,y}(x)|^2 + \frac{C}{\alpha}(|u(s)|^2 + |y(s)|^2)]ds, \end{aligned}$$

for any $\alpha > 0$. Combining these inequalities we get

$$p_t(x) \leq -(\varepsilon - \eta - \alpha)|\Phi_{s,t}^{u,y}(x)|^2 + C(u,y) \tag{38}$$

where $C(u,y) \geq 0$ is a constant depending on the inputs u, y, and on η, α. Select η, α so that $\varepsilon - \eta - \alpha > 0$. Then if $x \notin M_{as}$ we have $\Phi_{s,t}^{u,y}(x) \to \infty$ as $t \to \infty$, and hence from (38) we get $\limsup_{t\to\infty} p_t(x) \leq -\infty$.

We can now conclude that $\delta_{M_{as}} + \phi^{u,y}$ is the limit of p_t as $t \to \infty$. □

Conjecture 4.7 *The previous Theorem 4.6 holds for all functions u, y in L^2 (or at least decaying reasonably) and gives (fairly strong) convergence of p_t as in (33).*

5 Information State Controllers

A rich class of controllers have dynamics acting on real valued functions p on state space $\mathbf{R}^n$ and have output which is a function $u^*(p)$ which (i) is continuous in an appropriate topology on p, and (ii) satisfies $u^*(p+$ constant function $) = u^*(p)$. We call such a controller an *information state controller* (IS controller) provided its dynamics are given by the information state equation

$$\dot{p}_t = F(p_t, u^*(p), y(t)), \tag{39}$$

with a specified initial function p_0. Call the controller *null initialized* provided when the input function y is identically zero, the output is identically zero. Certainly this is a desirable property.

When H_∞ control theory is extended to nonlinear systems one finds under hypotheses we do not detail here (see [9,?]) that *if an acceptable controller K exists solving the H_∞ control problem, a certain type of information state controller $K^*_{p_0}$ called a "central controller" exists with finite initializing function p_0 which also solves the H_∞ control problem.* The main motivation for studying information state controllers as opposed to central controllers is that there is much we can say without knowing more detailed properties of u^*.

The results of this article have obvious implications for information state controllers.

> *Suppose Conjecture 4.7 is true. If a null initialized information state controller $K^*_{p_0}$ with p_0 finite solves the H_∞ control problem, then (i) a unique (up to a constant) equilibrium state solution p_e to (8) exists, (ii) for any initial condition x_0 and p_0 finite, all p_t in the closed loop system $(G, K^*_{p_0})$ resulting from any "rapidly decaying" input function w converges to $p_e +$ const, and (iii) $K^*_{p_e}$ is null initialized.*

A Appendix

Lemma A.1 *Let $\Phi^v_{s,t}(x)$ be the semigroup corresponding to*

$$\dot{x}(s) = f(x(s)) + gv(s), \quad x(t) = x, \tag{40}$$

with inputs $v \in L_2[0,\infty)$. Assume that f is strongly antistable (i.e. (22) holds).

1. *Then there exist constants $C > 0$, $c > 0$ such that*

$$|\Phi^v_{s,t}(x)| \leq Ce^{-c(t-s)}|x| + (\int_s^t e^{-c(r-s)}|v(r)|^2 dr)^{\frac{1}{2}}, \tag{41}$$

2. $$\int_T^t |\Phi^v_{s,t}(x)|^2 ds \leq C|x|^2 + C\int_T^t |v(s)|^2 ds. \tag{42}$$

3. $$|\Phi^{v_1}_{s,t}(x_1) - \Phi^{v_2}_{s,t}(x_2)| \leq C|x_1 - x_2|e^{-c(t-s)} + \; (\int_s^t e^{-c(r-s)}|v_1(r) - v_2(r)|^2 dr)^{\frac{1}{2}}, \tag{43}$$

4. $$|\nabla_x \Phi^v_{s,t}(x)| \leq Ce^{-c(t-s)}. \tag{44}$$

PROOF. Part (i). Let $L(x) = \frac{1}{2}|x|^2$ be a Lyapunov function for $-f$. Fix $x \in \mathbf{R}^n$, $t > s$ and write $\xi(s) = \Phi^v_{s,t}(x)$. Then ($C > 0$ and $c > 0$ will denote generic constants)

$$\begin{aligned}
\tfrac{d}{ds}L(\xi(s)) &= \nabla_x L(\xi(s))(f(\xi(s)) + g(\xi(s))v(s)) \\
&\geq cL(\xi(s)) - C|\nabla_x L(\xi(s))||v(s)| \\
&\geq c|\xi(s)|^2 - C\varepsilon|\xi(s)|^2 - \tfrac{C}{\varepsilon}|v(s)|^2 \\
&\geq cL(\xi(s)) - C|v(s)|^2
\end{aligned}$$

Using an integrating factor and integrating we get,

$$e^{-ct}L(x) - e^{-cs}L(\xi(s)) \geq -\int_s^t Ce^{-cr}|v(r)|^2 dr$$

and hence

$$L(\xi(s)) \leq e^{-c(t-s)}L(x) + \int_s^t Ce^{-c(r-s)}|v(r)|^2 dr,$$

which is enough to prove (41).

Part (ii). The second inequality (42) follows by integration, where it is noted that

$$\int_T^t \int_s^t Ce^{-c(r-s)}|v(r)|^2 dr ds \leq C\int_T^t |v(r)|^2 dr$$

(this can be seen by changing the order of integration).

Part (iii). Let $\zeta(s) = \Phi^{v_1}_{s,t}(x_1) - \Phi^{v_2}_{s,t}(x_2)$. Then using the stronger assumptions, we have

$$\begin{aligned}
\tfrac{d}{ds}\tfrac{1}{2}|\zeta(s)| &= \zeta(s)'(f(\Phi^{v_1}_{s,t}(x_1)) - f(\Phi^{v_2}_{s,t}(x_2)) + g(v_1(s) - v_2(s))) \\
&\geq c|\zeta(s)|^2 - C|\zeta(s)||v_1(s) - v_2(s)| \\
&\geq \tfrac{1}{2}c|\zeta(s)|^2 - C|v_1(s) - v_2(s)|^2
\end{aligned}$$

which is enough to prove (43).

Part (iv). Let $\zeta(s) = \nabla_x \Phi^v_{s,t}(x)$. Then

$$\frac{d}{ds}\tfrac{1}{2}|\zeta(s)|^2 = \zeta(s)'\nabla_x f(\Phi^v_{s,t}(x))\zeta(s) \geq \tfrac{1}{2}c|\zeta(s)|^2,$$

which proves (44). $\square$

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